

Supplementary Appendix to:
Perpetual Learning and Apparent Long Memory

Guillaume Chevillon
ESSEC Business School & CREST

Sophocles Mavroeidis
University of Oxford

April 13, 2017

Contents

1	Proof of Lemma 6	2
1.1	Preliminary results	2
1.1.1	Proof of Lemma S.1	2
1.1.2	Proof of Lemma S.2	3
1.2	Proof of Lemma 6	5
2	Proof of Lemma 7	7
3	Proof of Lemma 8	7

1 Proof of Lemma 6

1.1 Preliminary results

To derive our results we first need the following lemmas, S.1 and S.2.

Lemma S.1. *Let f a spectral density with f, f' and f'' bounded, $f > 0$ in a neighborhood of the origin and $f'(0) = 0$. Let $\delta \in (0, 1)$ and define the sequence*

$$a_j = \begin{cases} f\left(\frac{2\pi}{T}j\right)j^{-\delta}, & j \leq T; \\ f(0)j^{-\delta}, & j > T. \end{cases} \quad (\text{S-1})$$

Then $\{a_j\}$ is of pure bounded variation, defined as $\sum_{j=n}^{\infty} |\Delta a_j| = O(a_n)$ as $n \rightarrow \infty$ (Yong, 1974, Definition I-4).

Lemma S.2. *For $\delta \in (0, 1)$, as $(T\omega, \omega^{-1}) \rightarrow (\infty, \infty)$*

$$\begin{aligned} \sum_{j=1}^T a_j \cos j\omega &\sim \frac{\pi f(0)}{2\Gamma(\delta) \cos \frac{\delta\pi}{2}} \omega^{\delta-1}; \\ \sum_{j=1}^T a_j \sin j\omega &\sim \frac{\pi f(0)}{2\Gamma(\delta) \sin \frac{\delta\pi}{2}} \omega^{\delta-1}, \end{aligned}$$

where $\{a_j\}$ is defined in Lemma S.1.

1.1.1 Proof of Lemma S.1

When $j \leq T$, the difference Δa_j can be decomposed into

$$\Delta a_j = \frac{f(\omega_j) \left[(j-1)^\delta - j^\delta \right]}{j^\delta (j-1)^\delta} + \frac{f(\omega_j) - f(\omega_{j-1})}{(j-1)^\delta},$$

where a Taylor expansion yields

$$f(\omega_j) - f(\omega_{j-1}) = \frac{1}{2} f''(\omega_{j-1}) (\Delta\omega_j)^2 + o\left((\Delta\omega_j)^2\right),$$

and

$$\frac{f(\omega_j) \left[(j-1)^\delta - j^\delta \right]}{j^\delta (j-1)^\delta} = f(\omega_j) \frac{\left[(1-1/j)^\delta - 1 \right]}{(j-1)^\delta} = f(\omega_j) \frac{-\delta}{j^{1+\delta}} + o\left(\frac{1}{j^{1+\delta}}\right).$$

Since f and f'' are bounded, there exist $m_0, m_2 > 0$ such that

$$|\Delta a_j| \leq \frac{m_0 \delta}{j^{1+\delta}} + \frac{m_2}{2j^\delta} \left(\frac{2\pi}{T}\right)^2.$$

When $j > T$, the latter expression also holds with $m_2 = 0$ so for all j :

$$|\Delta a_j| \leq \frac{m_0 \delta}{j^{1+\delta}} + \frac{m_2}{2j^\delta} \left(\frac{2\pi}{T} \right)^2 1_{\{j \leq T\}},$$

where $1_{\{\cdot\}}$ is the indicator function.

Now consider $N \geq T \geq n$, then

$$\sum_{j=n}^N |\Delta a_j| = \sum_{j=n}^T |\Delta a_j| + \sum_{j=T+1}^N |\Delta a_j|, \quad (\text{S-2})$$

since for $j \leq T$, $T^{-2} \leq j^{-2}$, the previous expression rewrites as

$$\begin{aligned} \sum_{j=n}^N |\Delta a_j| &\leq \sum_{j=n}^T \left(\frac{m_0 \delta}{j^{1+\delta}} + \frac{2\pi^2 m_2}{j^{2+\delta}} \right) + \sum_{j=T+1}^N |\Delta a_j| \\ &\leq m_0 \left[n^{-\delta} - (N+1)^{-\delta} \right] + \frac{2\pi^2 m_2}{1+\delta} \left[n^{-1-\delta} - (T+1)^{-1-\delta} \right]. \end{aligned}$$

Letting $N \rightarrow \infty$, there exists M such that

$$\sum_{j=n}^{\infty} |\Delta a_j| \leq n^{-\delta} \left[m_0 + \frac{2\pi^2 m_2}{1+\delta} \frac{1}{n} \right] + \frac{M}{T^{\delta+1}},$$

and then if $a_n \neq 0$,

$$\frac{\sum_{j=n}^{\infty} |\Delta a_j|}{a_n} \leq \frac{\left[m_0 + \frac{2\pi^2 m_2}{1+\delta} \right] + \frac{M}{n^{-\delta} T^{\delta+1}}}{f\left(\frac{2\pi}{T} \min\{n, T\}\right)}.$$

We now use the fact that f is bounded above zero in a neighborhood of the origin so for T large enough, $f\left(\frac{2\pi}{T} n\right) > 0$. Also for $n \leq T$, $n^{-\delta} T^{\delta+1} \rightarrow \infty$ as $n \rightarrow \infty$. Therefore, there exist (n_0, M_0) such that for all $n \geq n_0$ with $a_n \neq 0$ and for $T \geq n$, $\frac{\sum_{j=n}^{\infty} |\Delta a_{T,j}|}{a_n} \leq M_0$, i.e. $\sum_{j=n}^{\infty} |\Delta a_j| = O(a_n)$.

Now assume that $T \leq n$, the previous expressions yield $\sum_{j=n}^{\infty} |\Delta a_j| \leq f(0) n^{-\delta} = a_n$ so the results also hold.

1.1.2 Proof of Lemma S.2

Introduce the Dirichlet kernel:

$$D_T(\omega) = \frac{1}{2} + \sum_{j=1}^T \cos j\omega = \frac{\sin(T+1/2)\omega}{2 \sin \frac{\omega}{2}},$$

where $|D_T(\omega) - 1/2| \leq \pi\omega^{-1}$ for $0 < \omega \leq \pi$ (Yong, 1974, p.39). Then, for all $\omega > 0$ and $T < N$,

$$\sum_{j=T+1}^N a_j \cos j\omega = f(0) \sum_{j=T+1}^N j^{-\delta} \cos j\omega.$$

Then

$$\begin{aligned}
& \left| \sum_{j=T+1}^N j^{-\delta} \cos j\omega \right| \\
&= \left| \sum_{j=T}^{N-1} \left(j^{-\delta} - (j+1)^{-\delta} \right) \left(D_j(\omega) - \frac{1}{2} \right) + N^{-\delta} \left(D_N(\omega) - \frac{1}{2} \right) - T^{-\delta} \left(D_T(\omega) - \frac{1}{2} \right) \right| \\
&\leq \sum_{j=T}^{N-1} \left| j^{-\delta} - (j+1)^{-\delta} \right| \left| D_j(\omega) - \frac{1}{2} \right| + N^{-\delta} \left| D_N(\omega) - \frac{1}{2} \right| + T^{-\delta} \left| D_T(\omega) - \frac{1}{2} \right|. \quad (\text{S-3})
\end{aligned}$$

We notice that $j^{-\delta} - (j+1)^{-\delta} = j^{-\delta} \left(1 - (1+j^{-1})^{-\delta} \right) \in \left[j^{-\delta} \left(\delta j^{-1} - \frac{\delta(\delta+1)}{2j^2} \right), j^{-\delta} \delta j^{-1} \right]$ so $\left| j^{-\delta} - (j+1)^{-\delta} \right| \leq \delta j^{-\delta-1}$. It follows that

$$\begin{aligned}
\left| \sum_{j=T+1}^N j^{-\delta} \cos j\omega \right| &\leq \delta \pi \omega^{-1} \left(T^{-\delta} - N^{-\delta} \right) + N^{-\delta} \pi \omega^{-1} + T^{-\delta} \pi \omega^{-1} \\
&= (1+\delta) \pi \omega^{-1} T^{-\delta} + (1-\delta) \pi \omega^{-1} N^{-\delta}.
\end{aligned}$$

Hence letting $N \rightarrow \infty$, for all $T, \omega > 0$,

$$\left| \sum_{j=1}^T j^{-\delta} \cos j\omega - \sum_{j=1}^{\infty} j^{-\delta} \cos j\omega \right| \leq (1+\delta) \pi \omega^{-1} T^{-\delta}. \quad (\text{S-4})$$

Now, Yong (1974), Theorem III-17, states that for $\delta \in (0, 1)$ since $\{a_j\}$ is of pure bounded variation

$$\sum_{j=1}^{\infty} a_j \cos j\omega \underset{\omega \rightarrow 0^+}{\sim} \frac{\pi f(0)}{2\Gamma(\delta) \cos \frac{\delta\pi}{2}} \omega^{\delta-1}.$$

Hence, in expression (S-4), as $(T, \omega^{-1}) \rightarrow (\infty, \infty)$: $\omega^{-1} T^{-\delta} = o(\omega^{\delta-1})$ if $\omega^{-1} = o(T)$, in which case

$$\sum_{j=1}^T a_j \cos j\omega \underset{(T, \omega) \rightarrow (\infty, 0)}{\sim} \frac{\pi f(0)}{2\Gamma(\delta) \cos \frac{\delta\pi}{2}} \omega^{\delta-1}.$$

When the summation involves a sine function, the proof is similar: $\left| \sum_{j=T+1}^{\infty} j^{-\delta} \sin j\omega \right| < (1+\delta) \pi \omega^{-1} T^{-\delta}$ using the conjugate Dirichlet kernel:

$$\overline{D}_T(\omega) = \sum_{j=1}^T \sin j\omega = \frac{\cos \frac{\omega}{2} - \cos \left(T + \frac{1}{2} \right) \omega}{2 \sin \frac{\omega}{2}},$$

with $|\overline{D}_T(\omega)| \leq \pi \omega^{-1}$ (also Yong, 1974, p. 39). Then an equivalent of expression (S-4) holds for the summation involving sine functions and we also refer to Theorem III-17 of Yong.

1.2 Proof of Lemma 6

First, if $k = 0$, $\sum_{j=1}^T j^{-\delta} f(\omega_j) = O(T^{1-\delta})$ since $f(\omega_T) = f(0) \neq 0$ so for $\delta \neq 1$ the results follows.

We now assume $k \neq 0$. Lemma S.2 implies as $(k, T/k) \rightarrow (\infty, \infty)$ that for $\delta \in (0, 1)$,

$$T^\delta \sum_{j=1}^T \frac{f(\omega_j) \cos(j\omega_k)}{j^\delta} \sim \frac{\pi f(0)}{2\Gamma(\delta) \cos \frac{\delta\pi}{2}} \left(2\pi \frac{k}{T}\right)^{\delta-1} T^\delta. \quad (\text{S-5})$$

Now, let $\delta \in (-1, 0)$. We use a procedure similar to integration by part to express (S-5) in terms of $\sum_{j=1}^T j^{-(\delta+1)} f_x(\omega_j) \sin(j\omega_k)$ where $\delta + 1 \in (0, 1)$ which allows to work alongside the previous results. Start with

$$\begin{aligned} & (j+1)^{-\delta} \sin((j+1)\omega_k) - j^{-\delta} \sin(j\omega_k) \\ &= \left((j+1)^{-\delta} - j^{-\delta} \right) \sin((j+1)\omega_k) + j^{-\delta} \{ \sin((j+1)\omega_k) - \sin(j\omega_k) \}, \end{aligned}$$

where $(j+1)^{-\delta} - j^{-\delta} = -\delta(j+1)^{-\delta-1} + o(j^{-\delta-1})$ and

$$\sin((j+1)\omega_k) - \sin(j\omega_k) = \sin(j\omega_k) (\cos(\omega_k) - 1) + \cos(j\omega_k) \sin(\omega_k).$$

Hence

$$\begin{aligned} & \sum_{j=1}^T \left[(j+1)^{-\delta} \sin((j+1)\omega_k) - j^{-\delta} \sin(j\omega_k) \right] \\ &= \sum_{j=1}^T \left\{ -\delta(j+1)^{-\delta-1} + o\left((j+1)^{-\delta-1}\right) \right\} \sin((j+1)\omega_k) \\ &+ \sin(\omega_k) \sum_{j=1}^T j^{-\delta} \cos(j\omega_k) + (\cos(\omega_k) - 1) \sum_{j=1}^T j^{-\delta} \sin(j\omega_k), \end{aligned}$$

i.e. since $k \neq 0$:

$$\begin{aligned} \sum_{j=1}^T j^{-\delta} \cos(j\omega_k) &= (T+1)^{-\delta} \frac{\sin(T+1)\omega_k}{\sin(\omega_k)} - 1 - \delta \\ &- \frac{\cos(\omega_k) - 1}{\sin(\omega_k)} \sum_{j=1}^T j^{-\delta} \sin(j\omega_k) + \frac{\delta}{\sin(\omega_k)} \sum_{j=1}^{T+1} j^{-(1+\delta)} \sin(j\omega_k) \\ &+ o\left(\frac{\sum_{j=2}^{T+1} j^{-(1+\delta)} \sin(j\omega_k)}{\sin(\omega_k)} \right). \end{aligned}$$

Similarly, since $\cos((j+1)\omega_k) - \cos(j\omega_k) = \cos(j\omega_k) (\cos(\omega_k) - 1) - \sin(j\omega_k) \sin \omega_k$, we can express:

$$\begin{aligned}
\sum_{j=1}^T j^{-\delta} \sin(j\omega_k) &= -(T+1)^{-\delta} \frac{\cos(T+1)\omega_k}{\sin(\omega_k)} + (1+\delta) \frac{\cos\omega_k}{\sin\omega_k} \\
&+ \frac{\cos(\omega_k) - 1}{\sin(\omega_k)} \sum_{j=1}^T j^{-\delta} \cos(j\omega_k) - \frac{\delta}{\sin(\omega_k)} \sum_{j=1}^{T+1} j^{-(1+\delta)} \cos(j\omega_k) \\
&+ o\left(\frac{\sum_{j=2}^{T+1} j^{-(1+\delta)} \cos(j\omega_k)}{\sin(\omega_k)}\right).
\end{aligned}$$

Plugging $\sum_{j=1}^T j^{-\delta} \sin(j\omega_k)$ in the expression for $\sum_{j=1}^T j^{-\delta} \cos(j\omega_k)$ yields:

$$\begin{aligned}
&\left(1 + \frac{(\cos(\omega_k) - 1)^2}{\sin^2(\omega_k)}\right) \sum_{j=1}^T j^{-\delta} \cos(j\omega_k) \\
&= (T+1)^{-\delta} \frac{\sin(\omega_k) \sin[(T+1)\omega_k] + (\cos(\omega_k) - 1) \cos[(T+1)\omega_k]}{\sin^2(\omega_k)} \\
&- (1+\delta) \left(1 + \frac{\cos(\omega_k) - 1}{\sin^2(\omega_k)} \cos\omega_k\right) \\
&+ \frac{\delta}{\sin(\omega_k)} \sum_{j=1}^{T+1} j^{-(1+\delta)} \sin(j\omega_k) + \delta \frac{\cos(\omega_k) - 1}{\sin^2(\omega_k)} \sum_{j=1}^{T+1} j^{-(1+\delta)} \cos(j\omega_k) \\
&+ o\left(\frac{\sum_{j=2}^{T+1} j^{-(1+\delta)} \sin(j\omega_k)}{\sin(\omega_k)}\right) + o\left(\frac{\cos(\omega_k) - 1}{\sin^2(\omega_k)} \sum_{j=2}^{T+1} j^{-(1+\delta)} \cos(j\omega_k)\right).
\end{aligned}$$

Using Lemma S.2, this leads, as $(T\omega_k, \omega_k^{-1}) \rightarrow (\infty, \infty)$, and since $1 + \delta \in (0, 1)$ to:

$$\begin{aligned}
\sum_{j=1}^T j^{-\delta} \cos(j\omega_k) &= \frac{1}{T^\delta \omega_k} \sin[(T+1)\omega_k] - (1+\delta) \left(1 + \frac{1}{2}\right) \\
&+ \frac{\delta \pi f(0)}{2\Gamma(\delta+1) \cos \frac{(\delta+1)\pi}{2}} \omega_k^{\delta-1} \\
&+ o\left(\omega_k^{\delta-1}\right),
\end{aligned}$$

where $T^{-\delta} \omega_k^{-1} = O\left(\omega_k^{\delta-1} k^{-\delta}\right)$. Hence

$$\sum_{j=1}^T j^{-\delta} \cos(j\omega_k) = \frac{\delta \pi f(0)}{2\Gamma(\delta+1) \cos \frac{(\delta+1)\pi}{2}} \omega_k^{\delta-1} + o\left(\omega_k^{\delta-1}\right), \tag{S-6}$$

which simplifies to $T^{\delta-1} \sum_{j=1}^T j^{-\delta} \cos(j\omega_k) \asymp k^{\delta-1}$.

2 Proof of Lemma 7

For $\delta_\kappa \in (0, 1)$, $\delta_\kappa - 2 \in (-2, -1)$. Yong (1974), Theorems III-24 and -27, show that for $\delta_\kappa \in (0, 1)$, if there exists a function S slowly varying at infinity such that $a_j = j^{\delta_\kappa - 2} S(j)$, then

$$\begin{aligned} \sum_{j=1}^{\infty} a_j \cos(j\omega) - \sum_{j=1}^{\infty} a_j &\underset{\omega \rightarrow 0^+}{\sim} \frac{\pi}{2\Gamma(2 - \delta_\kappa) \cos \frac{(2 - \delta_\kappa)\pi}{2}} \omega^{1 - \delta_\kappa} S\left(\frac{1}{\omega}\right) \\ \sum_{j=1}^{\infty} a_j \sin(j\omega) &\underset{\omega \rightarrow 0^+}{\sim} \frac{\pi}{2\Gamma(2 - \delta_\kappa) \sin \frac{(2 - \delta_\kappa)\pi}{2}} \omega^{1 - \delta_\kappa} S\left(\frac{1}{\omega}\right). \end{aligned}$$

Define, for $x \geq 1$, $S(x) = \kappa_{\lfloor x \rfloor} / \lfloor x \rfloor^{\delta_\kappa - 2}$, where $\lfloor x \rfloor$ is the integer part of x . Then as $x \rightarrow \infty$ and for $\delta \geq 1/x$,

$$S(\delta x) / S(x) = \frac{\kappa_{\lfloor \delta x \rfloor} \lfloor x \rfloor^{\delta_\kappa - 2}}{\kappa_{\lfloor x \rfloor} \lfloor \lambda x \rfloor^{\delta_\kappa - 2}} \rightarrow 1,$$

so S is slowly varying with $S\left(\frac{1}{\omega}\right) \rightarrow c_\kappa$ as $\omega \rightarrow 0$. This implies, using $\kappa_j = j^{\delta_\kappa - 2} S(j)$ for $j \geq 1$ and $\sum_{j=0}^{\infty} \kappa_j = \kappa(1) = 1$, that

$$\kappa(e^{-i\omega}) - 1 \underset{\omega \rightarrow 0^+}{\sim} \frac{\pi c_\kappa}{2\Gamma(2 - \delta_\kappa)} \left[-\frac{1}{\cos \frac{\pi \delta_\kappa}{2}} + i \frac{1}{\sin \frac{\pi \delta_\kappa}{2}} \right] \omega^{1 - \delta_\kappa},$$

i.e. the result holds for $\text{Re}(\kappa(e^{i\omega}) - 1)$ setting $c_\kappa^* = \frac{\pi c_\kappa}{2\Gamma(2 - \delta_\kappa) \cos \frac{\pi \delta_\kappa}{2}}$ such that $c_\kappa c_\kappa^* > 0$. Also, using

$$\left| \frac{-1}{\cos \frac{\pi \delta_\kappa}{2}} + i \frac{1}{\sin \frac{\pi \delta_\kappa}{2}} \right|^2 = \left(\cos \frac{\pi \delta_\kappa}{2} \sin \frac{\pi \delta_\kappa}{2} \right)^{-2} = \left(\frac{\sin \pi \delta_\kappa}{2} \right)^{-2},$$

and $\Gamma(1 + z) = z\Gamma(z)$ together with $\Gamma(1 - z)\Gamma(z) = \frac{\pi}{\sin \pi z}$, we obtain

$$|1 - \kappa(e^{i\omega})|^2 \sim \frac{c_\kappa^2 \Gamma(\delta_\kappa)^2}{(1 - \delta_\kappa)^2} \omega^{2(1 - \delta_\kappa)}, \quad (\text{S-7})$$

and $c_\kappa^{**} = \frac{c_\kappa^2 \Gamma(\delta_\kappa)^2}{(1 - \delta_\kappa)^2} > 0$.

3 Proof of Lemma 8

Consider

$$\begin{aligned} f_y(\omega) - f_y(0) &= \frac{(f_x(\omega) - f_x(0))(1 - \beta)^2}{(1 - \beta)^2 |1 - \beta + \beta(1 - \kappa(e^{-i\omega}))|^2} \\ &\quad - f_x(0) \frac{2\beta(1 - \beta) \text{Re}[1 - \kappa(e^{-i\omega})] + \beta^2 |1 - \kappa(e^{-i\omega})|^2}{(1 - \beta)^2 |1 - \beta + \beta(1 - \kappa(e^{-i\omega}))|^2}, \end{aligned}$$

since

$$|1 - \beta + \beta(1 - \kappa(e^{-i\omega}))|^2 = (1 - \beta)^2 - 2\beta(1 - \beta) \operatorname{Re}[\kappa(e^{-i\omega}) - 1] - \beta^2 |\kappa(e^{-i\omega}) - 1|^2.$$

Under Assumption B, $|f'_x(0)| < \infty$ (see Stock, 1994), so $f_x(\omega) - f_x(0) = O(\omega)$. Now if $\delta_\kappa > 0$, under constant learning, $\kappa_j \sim c_\kappa j^{\delta_\kappa - 2}$ for some $c_\kappa \neq 0$. Lemma 6 implies that there exist $c_\kappa^* \neq 0$, with $c_\kappa c_\kappa^* > 0$, and $c_\kappa^{**} > 0$ such that

$$\begin{aligned} f_y(\omega) - f_y(0) &\underset{\omega \rightarrow 0^+}{\sim} \frac{-2\beta(1 - \beta)f_x(0)c_\kappa^* \omega^{1 - \delta_\kappa} + \beta^2 f_x(0)c_\kappa^{**} \omega^{2(1 - \delta_\kappa)}}{(1 - \beta)^4} \\ &\underset{\omega \rightarrow 0^+}{\sim} -\frac{2\beta f_x(0)c_\kappa^*}{(1 - \beta)^3} \omega^{1 - \delta_\kappa}. \end{aligned} \quad (\text{S-8})$$

We first note that by definition of the population spectrum

$$f_y(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k e^{-i\omega k} = \frac{1}{2\pi} \left\{ \gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k \cos \omega k \right\},$$

where $\gamma_k = \operatorname{Cov}(y_t, y_{t-k})$, is symmetric since y_t is stationary. We assume for now that γ_k is of bounded variation, then, for $\omega \neq 0$, the series $\sum_{k=1}^n \gamma_k \cos \omega k$ converges uniformly as $n \rightarrow \infty$ (see Zygmund, 1935, Section. 1.23). It follows that the derivative of f_y satisfies:

$$f'_y(\omega) = -\frac{1}{\pi} \sum_{k=1}^{\infty} k \gamma_k \sin k\omega. \quad (\text{S-9})$$

We now use Theorem III-11 of Yong (1974) who works under the assumption that $\{a_k\}_{k \in \mathbb{N}}$ is a sequence of positive numbers that is quasi-monotonically convergent to zero (i.e. $a_k \rightarrow 0$ and there exist $M \geq 0$, such that $a_{k+1} \leq a_k(1 + \frac{M}{k})$ for all $k \geq k_0(M)$) and that $\{a_k\}$ is also of bounded variation, i.e. $\sum_{k=1}^{\infty} |\Delta a_k| < \infty$. The theorem states that for $a \in (0, 1)$, $a_k \sim k^{-a} S^*(k)$ as $k \rightarrow \infty$, with S^* slowly varying, if and only if

$$\sum_{k=1}^{\infty} a_k \sin k\omega \sim \frac{\pi}{2\Gamma(a) \sin \frac{\pi a}{2}} \omega^{a-1} S^*\left(\frac{1}{\omega}\right) \text{ as } \omega \rightarrow 0^+.$$

We apply this theorem to (S-9), using expression (S-8) that $f'_y(\omega) \underset{\omega \rightarrow 0^+}{\sim} -\frac{2\beta f_x(0)c_\kappa^*}{(1-\beta)^3} \omega^{-\delta_\kappa}$

$$-\frac{1}{\pi} \sum_{k=1}^{\infty} k \gamma_k \sin k\omega \underset{\omega \rightarrow 0^+}{\sim} -\frac{2\beta f_x(0)c_\kappa^*}{(1-\beta)^3} \omega^{-\delta_\kappa}.$$

We let $a = 1 - \delta_\kappa$ in the theorem of Yong above, defining

$$a_k = \frac{k \gamma_k}{\pi} \frac{(1 - \beta)^3}{2\beta f_x(0) c_\kappa^*}.$$

This implies that $a_k \sim \frac{\pi}{2\Gamma(a)\sin\frac{\pi a}{2}} k^{-(1-\delta_\kappa)}$, with $\Gamma(1-\delta_\kappa)\sin\frac{\pi(1-\delta_\kappa)}{2} = \frac{\pi}{2\Gamma(\delta_\kappa)\sin\frac{\pi\delta_\kappa}{2}}$, i.e.

$$\gamma_k \sim \frac{2\pi\beta f_x(0) c_\kappa^* \Gamma(\delta_\kappa) \sin\frac{\pi\delta_\kappa}{2}}{(1-\beta)^3} k^{-(2-\delta_\kappa)}. \quad (\text{S-10})$$

To apply Theorem III-11 of Yong (1974), we check that a_k thus defined is a quasi-monotonic sequence with bounded variation. The first holds since $a_{k+1}/a_k \sim (1+1/k)^{-(1-\delta_\kappa)} < 1$, so choose M such that $a_{k+1}/a_k < 1$ for $k > M$. Also $k\gamma_k$ is clearly of bounded variation since it is asymptotically positive and

$$\Delta(k\gamma_k) = O\left(k^{-(2-\delta_\kappa)}\right)$$

is summable. Finally, we check the uniform convergence condition in Zygmund (1935): $|\Delta\gamma_k| = O\left(k^{-(3-\delta_\kappa)}\right)$ so γ_k is of bounded variation.

References

- Yong, C. H. (1974). *Asymptotic Behaviour of Trigonometric Series*. Chinese University of Hong Kong.
- Zygmund, A. (1935). *Trigonometrical Series*. Warsaw: Monografie Matematyczne V.