Supplementary Appendix to: Perpetual Learning and Apparent Long Memory

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1.1 Preliminary results

To derive our results we first need the following lemmas, S.1 and S.2.

Lemma S.1. *Let f a spectral density with f, f' and f'' bounded, $f > 0$ in a neighborhood of the origin and $f'(0) = 0$. Let $\delta \in (0,1)$ and define the sequence*

$$a_j = \begin{cases} f\left(\frac{2\pi}{T} j\right) j^{-\delta}, & j \le T; \\ f(0)\, j^{-\delta}, & j > T. \end{cases} \tag{S–1}$$

Then $\{a_j\}$ is of pure bounded variation, defined as $\sum_{j=n}^{\infty} |\Delta a_j| = O(a_n)$ as $n \to \infty$ (Yong, 1974, Definition I-4).

Lemma S.2. *For $\delta \in (0,1)$, as $\left(T\omega, \omega^{-1}\right) \to (\infty, \infty)$*

$$\sum_{j=1}^{T} a_j \cos j\omega \sim \frac{\pi f(0)}{2\Gamma(\delta)\cos\frac{\delta\pi}{2}} \omega^{\delta-1};$$

$$\sum_{j=1}^{T} a_j \sin j\omega \sim \frac{\pi f(0)}{2\Gamma(\delta)\sin\frac{\delta\pi}{2}} \omega^{\delta-1},$$

where $\{a_j\}$ is defined in Lemma S.1.

1.1.1 Proof of Lemma S.1

When $j \le T$, the difference Δa_j can be decomposed into

$$\Delta a_j = \frac{f(\omega_j)\left[(j-1)^{\delta} - j^{\delta}\right]}{j^{\delta}(j-1)^{\delta}} + \frac{f(\omega_j) - f(\omega_{j-1})}{(j-1)^{\delta}},$$

where a Taylor expansion yields

$$f(\omega_j) - f(\omega_{j-1}) = \frac{1}{2} f''(\omega_{j-1})(\Delta\omega_j)^2 + o\left((\Delta\omega_j)^2\right),$$

and

$$\frac{f(\omega_j)\left[(j-1)^{\delta} - j^{\delta}\right]}{j^{\delta}(j-1)^{\delta}} = f(\omega_j)\frac{\left[(1-1/j)^{\delta} - 1\right]}{(j-1)^{\delta}} = f(\omega_j)\frac{-\delta}{j^{1+\delta}} + o\left(\frac{1}{j^{1+\delta}}\right).$$

Since f and f'' are bounded, there exist $m_0, m_2 >$ such that

$$|\Delta a_j| \le \frac{m_0\delta}{j^{1+\delta}} + \frac{m_2}{2j^{\delta}}\left(\frac{2\pi}{T}\right)^2.$$

When $j > T$, the latter expression also holds with $m_2 = 0$ so for all j :

$$|\Delta a_j| \leq \frac{m_0 \delta}{j^{1+\delta}} + \frac{m_2}{2j^\delta}\left(\frac{2\pi}{T}\right)^2 1_{\{j\leq T\}},$$

where $1_{\{.\}}$ is the indicator function.

Now consider $N \geq T \geq n$, then

$$\sum_{j=n}^{N} |\Delta a_j| = \sum_{j=n}^{T} |\Delta a_j| + \sum_{j=T+1}^{N} |\Delta a_j|, \tag{S–2}$$

since for $j \leq T$, $T^{-2} \leq j^{-2}$, the previous expression rewrites as

$$\begin{aligned}\sum_{j=n}^{N} |\Delta a_j| &\leq \sum_{j=n}^{T}\left(\frac{m_0\delta}{j^{1+\delta}} + \frac{2\pi^2 m_2}{j^{2+\delta}}\right) + \sum_{j=T+1}^{N} |\Delta a_j| \\ &\leq m_0\left[n^{-\delta} - (N+1)^{-\delta}\right] + \frac{2\pi^2 m_2}{1+\delta}\left[n^{-1-\delta} - (T+1)^{-1-\delta}\right].\end{aligned}$$

Letting $N \to \infty$, there exists M such that

$$\sum_{j=n}^{\infty} |\Delta a_j| \leq n^{-\delta}\left[m_0 + \frac{2\pi^2 m_2}{1+\delta}\frac{1}{n}\right] + \frac{M}{T^{\delta+1}},$$

and then if $a_n \neq 0$,

$$\frac{\sum_{j=n}^{\infty} |\Delta a_j|}{a_n} \leq \frac{\left[m_0 + \frac{2\pi^2 m_2}{1+\delta}\right] + \frac{M}{n^{-\delta} T^{\delta+1}}}{f\left(\frac{2\pi}{T}\min\{n, T\}\right)}.$$

We now use the fact that f is bounded above zero in a neighborhood of the origin so for T large enough, $f\left(\frac{2\pi}{T}n\right) > 0$. Also for $n \leq T$, $n^{-\delta}T^{\delta+1} \to \infty$ as $n \to \infty$. Therefore, there exist (n_0, M_0) such that for all $n \geq n_0$ with $a_n \neq 0$ and for $T \geq n$, $\frac{\sum_{j=n}^{\infty}|\Delta a_{T,j}|}{a_n} \leq M_0$, i.e. $\sum_{j=n}^{\infty} |\Delta a_j| = O(a_n)$.

Now assume that $T \leq n$, the previous expressions yield $\sum_{j=n}^{\infty} |\Delta a_j| \leq f(0)\, n^{-\delta} = a_n$ so the results also hold.

1.1.2 Proof of Lemma S.2

Introduce the Dirichlet kernel:

$$D_T(\omega) = \frac{1}{2} + \sum_{j=1}^{T} \cos j\omega = \frac{\sin(T+1/2)\,\omega}{2\sin\frac{\omega}{2}},$$

where $|D_T(\omega) - 1/2| \leq \pi\omega^{-1}$ for $0 < \omega \leq \pi$ (Yong, 1974, p.39). Then, for all $\omega > 0$ and $T < N$,

$$\sum_{j=T+1}^{N} a_j \cos j\omega = f(0) \sum_{j=T+1}^{N} j^{-\delta} \cos j\omega.$$

Then

$$\left|\sum_{j=T+1}^{N} j^{-\delta}\cos j\omega\right|$$
$$= \left|\sum_{j=T}^{N-1}\left(j^{-\delta}-(j+1)^{-\delta}\right)\left(D_j(\omega)-\frac{1}{2}\right)+N^{-\delta}\left(D_N(\omega)-\frac{1}{2}\right)-T^{-\delta}\left(D_T(\omega)-\frac{1}{2}\right)\right|$$
$$\leq \sum_{j=T}^{N-1}\left|j^{-\delta}-(j+1)^{-\delta}\right|\left|D_j(\omega)-\frac{1}{2}\right|+N^{-\delta}\left|D_N(\omega)-\frac{1}{2}\right|+T^{-\delta}\left|D_T(\omega)-\frac{1}{2}\right|. \quad \text{(S–3)}$$

We notice that $j^{-\delta}-(j+1)^{-\delta}=j^{-\delta}\left(1-\left(1+j^{-1}\right)^{-\delta}\right)\in\left[j^{-\delta}\left(\delta j^{-1}-\frac{\delta(\delta+1)}{2j^2}\right),j^{-\delta}\delta j^{-1}\right]$ so $\left|j^{-\delta}-(j+1)^{-\delta}\right|\leq\delta j^{-\delta-1}$. It follows that

$$\left|\sum_{j=T+1}^{N} j^{-\delta}\cos j\omega\right|\leq\delta\pi\omega^{-1}\left(T^{-\delta}-N^{-\delta}\right)+N^{-\delta}\pi\omega^{-1}+T^{-\delta}\pi\omega^{-1}$$
$$=(1+\delta)\,\pi\omega^{-1}T^{-\delta}+(1-\delta)\,\pi\omega^{-1}N^{-\delta}.$$

Hence letting $N\to\infty$, for all $T,\omega>0$,

$$\left|\sum_{j=1}^{T} j^{-\delta}\cos j\omega-\sum_{j=1}^{\infty} j^{-\delta}\cos j\omega\right|\leq(1+\delta)\,\pi\omega^{-1}T^{-\delta}. \quad \text{(S–4)}$$

Now, Yong (1974), Theorem III-17, states that for $\delta\in(0,1)$ since $\{a_j\}$ is of pure bounded variation

$$\sum_{j=1}^{\infty} a_j\cos j\omega \underset{\omega\to0^+}{\sim}\frac{\pi f(0)}{2\Gamma(\delta)\cos\frac{\delta\pi}{2}}\omega^{\delta-1}.$$

Hence, in expression (S–4), as $\left(T,\omega^{-1}\right)\to(\infty,\infty)$: $\omega^{-1}T^{-\delta}=o\left(\omega^{\delta-1}\right)$ if $\omega^{-1}=o(T)$, in which case

$$\sum_{j=1}^{T} a_j\cos j\omega \underset{(T\omega,\omega)\to(\infty,0)}{\sim}\frac{\pi f(0)}{2\Gamma(\delta)\cos\frac{\delta\pi}{2}}\omega^{\delta-1}.$$

When the summation involves a sine function, the proof is similar: $\left|\sum_{j=T+1}^{\infty} j^{-\delta}\sin j\omega\right|<(1+\delta)\,\pi\omega^{-1}T^{-\delta}$ using the conjugate Dirichlet kernel:

$$\overline{D}_T(\omega)=\sum_{j=1}^{T}\sin j\omega=\frac{\cos\frac{\omega}{2}-\cos\left(T+\frac{1}{2}\right)\omega}{2\sin\frac{\omega}{2}},$$

with $\left|\overline{D}_T(\omega)\right|\leq\pi\omega^{-1}$ (also Yong, 1974, p. 39). Then an equivalent of expression (S–4) holds for the summation involving sine functions and we also refer to Theorem III-17 of Yong.

1.2 Proof of Lemma 6

First, if $k=0$, $\sum_{j=1}^{T} j^{-\delta} f(\omega_j) = O\left(T^{1-\delta}\right)$ since $f(\omega_T) = f(0) \neq 0$ so for $\delta \neq 1$ the results follows.

We now assume $k \neq 0$. Lemma S.2 implies as $(k, T/k) \to (\infty, \infty)$ that for $\delta \in (0,1)$,

$$T^{\delta} \sum_{j=1}^{T} \frac{f(\omega_j)\cos(j\omega_k)}{j^{\delta}} \sim \frac{\pi f(0)}{2\Gamma(\delta)\cos\frac{\delta\pi}{2}} \left(2\pi \frac{k}{T}\right)^{\delta-1} T^{\delta}. \tag{S–5}$$

Now, let $\delta \in (-1,0)$. We use a procedure similar to integration by part to express (S–5) in terms of $\sum_{j=1}^{T} j^{-(\delta+1)} f_x(\omega_j)\sin(j\omega_k)$ where $\delta+1 \in (0,1)$ which allows to work alongside the previous results. Start with

$$\begin{aligned}
&(j+1)^{-\delta}\sin((j+1)\omega_k) - j^{-\delta}\sin(j\omega_k) \\
&= \left((j+1)^{-\delta} - j^{-\delta}\right)\sin((j+1)\omega_k) + j^{-\delta}\left\{\sin((j+1)\omega_k) - \sin(j\omega_k)\right\},
\end{aligned}$$

where $(j+1)^{-\delta} - j^{-\delta} = -\delta(j+1)^{-\delta-1} + o\left(j^{-\delta-1}\right)$ and

$$\sin((j+1)\omega_k) - \sin(j\omega_k) = \sin(j\omega_k)(\cos(\omega_k)-1) + \cos(j\omega_k)\sin(\omega_k).$$

Hence

$$\begin{aligned}
&\sum_{j=1}^{T}\left[(j+1)^{-\delta}\sin((j+1)\omega_k) - j^{-\delta}\sin(j\omega_k)\right] \\
&= \sum_{j=1}^{T}\left\{-\delta(j+1)^{-\delta-1} + o\left((j+1)^{-\delta-1}\right)\right\}\sin((j+1)\omega_k) \\
&+ \sin(\omega_k)\sum_{j=1}^{T} j^{-\delta}\cos(j\omega_k) + (\cos(\omega_k)-1)\sum_{j=1}^{T} j^{-\delta}\sin(j\omega_k),
\end{aligned}$$

i.e. since $k \neq 0$:

$$\begin{aligned}
\sum_{j=1}^{T} j^{-\delta}\cos(j\omega_k) = (T+1)^{-\delta}\,&\frac{\sin(T+1)\omega_k}{\sin(\omega_k)} - 1 - \delta \\
&- \frac{\cos(\omega_k)-1}{\sin(\omega_k)}\sum_{j=1}^{T} j^{-\delta}\sin(j\omega_k) + \frac{\delta}{\sin(\omega_k)}\sum_{j=1}^{T+1} j^{-(1+\delta)}\sin(j\omega_k) \\
&+ o\left(\frac{\sum_{j=2}^{T+1} j^{-(1+\delta)}\sin(j\omega_k)}{\sin(\omega_k)}\right).
\end{aligned}$$

Similarly, since $\cos((j+1)\omega_k) - \cos(j\omega_k) = \cos(j\omega_k))(\cos(\omega_k)-1) - \sin(j\omega_k)\sin\omega_k$, we can express:

$$\sum_{j=1}^{T} j^{-\delta} \sin(j\omega_k) = -(T+1)^{-\delta} \frac{\cos(T+1)\omega_k}{\sin(\omega_k)} + (1+\delta)\frac{\cos\omega_k}{\sin\omega_k}$$
$$+ \frac{\cos(\omega_k)-1}{\sin(\omega_k)} \sum_{j=1}^{T} j^{-\delta}\cos(j\omega_k) - \frac{\delta}{\sin(\omega_k)} \sum_{j=1}^{T+1} j^{-(1+\delta)} \cos(j\omega_k)$$
$$+ o\left(\frac{\sum_{j=2}^{T+1} j^{-(1+\delta)}\cos(j\omega_k)}{\sin(\omega_k)}\right).$$

Plugging $\sum_{j=1}^{T} j^{-\delta}\sin(j\omega_k)$ in the expression for $\sum_{j=1}^{T} j^{-\delta}\cos(j\omega_k)$ yields:

$$\left(1 + \frac{(\cos(\omega_k)-1)^2}{\sin^2(\omega_k)}\right) \sum_{j=1}^{T} j^{-\delta}\cos(j\omega_k)$$
$$= (T+1)^{-\delta} \frac{\sin(\omega_k)\sin[(T+1)\omega_k] + (\cos(\omega_k)-1)\cos[(T+1)\omega_k]}{\sin^2(\omega_k)}$$
$$-(1+\delta)\left(1 + \frac{\cos(\omega_k)-1}{\sin^2(\omega_k)}\cos\omega_k\right)$$
$$+ \frac{\delta}{\sin(\omega_k)} \sum_{j=1}^{T+1} j^{-(1+\delta)}\sin(j\omega_k) + \delta\frac{\cos(\omega_k)-1}{\sin^2(\omega_k)} \sum_{j=1}^{T+1} j^{-(1+\delta)}\cos(j\omega_k)$$
$$+ o\left(\frac{\sum_{j=2}^{T+1} j^{-(1+\delta)}\sin(j\omega_k)}{\sin(\omega_k)}\right) + o\left(\frac{\cos(\omega_k)-1}{\sin^2(\omega_k)} \sum_{j=2}^{T+1} j^{-(1+\delta)}\cos(j\omega_k)\right).$$

Using Lemma S.2, this leads, as $\left(T\omega_k, \omega_k^{-1}\right) \to (\infty, \infty)$, and since $1+\delta \in (0,1)$ to:

$$\sum_{j=1}^{T} j^{-\delta}\cos(j\omega_k) = \frac{1}{T^{\delta}\omega_k}\sin[(T+1)\omega_k] - (1+\delta)\left(1+\frac{1}{2}\right)$$
$$+ \frac{\delta\pi f(0)}{2\Gamma(\delta+1)\cos\frac{(\delta+1)\pi}{2}}\omega_k^{\delta-1}$$
$$+ o\left(\omega_k^{\delta-1}\right),$$

where $T^{-\delta}\omega_k^{-1} = O\left(\omega_k^{\delta-1}k^{-\delta}\right)$. Hence

$$\sum_{j=1}^{T} j^{-\delta}\cos(j\omega_k) = \frac{\delta\pi f(0)}{2\Gamma(\delta+1)\cos\frac{(\delta+1)\pi}{2}}\omega_k^{\delta-1} + o\left(\omega_k^{\delta-1}\right), \tag{S–6}$$

which simplifies to $T^{\delta-1}\sum_{j=1}^{T} j^{-\delta}\cos(j\omega_k) \asymp k^{\delta-1}$.

2 Proof of Lemma 7

For $\delta_\kappa \in (0,1)$, $\delta_\kappa - 2 \in (-2,-1)$. Yong (1974), Theorems III-24 and -27, show that for $\delta_\kappa \in (0,1)$, if there exists a function S slowly varying at infinity such that $a_j = j^{\delta_\kappa - 2} S(j)$, then

$$\sum_{j=1}^{\infty} a_j \cos(j\omega) - \sum_{j=1}^{\infty} a_j \underset{\omega\to 0^+}{\sim} \frac{\pi}{2\Gamma(2-\delta_k)\cos\frac{(2-\delta_k)\pi}{2}} \omega^{1-\delta_\kappa} S\left(\frac{1}{\omega}\right)$$
$$\sum_{j=1}^{\infty} a_j \sin(j\omega) \underset{\omega\to 0^+}{\sim} \frac{\pi}{2\Gamma(2-\delta_k)\sin\frac{(2-\delta_k)\pi}{2}} \omega^{1-\delta_\kappa} S\left(\frac{1}{\omega}\right).$$

Define, for $x \geq 1$, $S(x) = \kappa_{\lfloor x \rfloor} / \lfloor x \rfloor^{\delta_\kappa - 2}$, where $\lfloor x \rfloor$ is the integer part of x. Then as $x \to \infty$ and for $\delta \geq 1/x$,

$$S(\delta x)/S(x) = \frac{\kappa_{\lfloor \delta x \rfloor}}{\kappa \lfloor x \rfloor} \frac{\lfloor x \rfloor^{\delta_\kappa - 2}}{\lfloor \lambda x \rfloor^{\delta_\kappa - 2}} \to 1,$$

so S is slowly varying with $S\left(\frac{1}{\omega}\right) \to c_\kappa$ as $\omega \to 0$. This implies, using $\kappa_j = j^{\delta_\kappa - 2} S(j)$ for $j \geq 1$ and $\sum_{j=0}^{\infty} \kappa_j = \kappa(1) = 1$, that

$$\kappa\left(e^{-i\omega}\right) - 1 \underset{\omega\to 0^+}{\sim} \frac{\pi c_\kappa}{2\Gamma(2-\delta_\kappa)} \left[-\frac{1}{\cos\frac{\pi\delta_\kappa}{2}} + i\frac{1}{\sin\frac{\pi\delta_\kappa}{2}} \right] \omega^{1-\delta_\kappa},$$

i.e. the result holds for $\mathrm{Re}\left(\kappa\left(e^{i\omega}\right) - 1\right)$ setting $c_\kappa^* = \frac{\pi c_\kappa}{2\Gamma(2-\delta_\kappa)} \frac{1}{\cos\frac{\pi\delta_\kappa}{2}}$ such that $c_\kappa c_\kappa^* > 0$. Also, using

$$\left| \frac{-1}{\cos\frac{\pi\delta_\kappa}{2}} + i\frac{1}{\sin\frac{\pi\delta_\kappa}{2}} \right|^2 = \left(\cos\frac{\pi\delta_\kappa}{2} \sin\frac{\pi\delta_\kappa}{2} \right)^{-2} = \left(\frac{\sin \pi\delta_\kappa}{2} \right)^{-2},$$

and $\Gamma(1+z) = z\Gamma(z)$ together with $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$, we obtain

$$\left|1 - \kappa\left(e^{i\omega}\right)\right|^2 \sim \frac{c_\kappa^2 \Gamma(\delta_\kappa)^2}{(1-\delta_\kappa)^2} \omega^{2(1-\delta_\kappa)}, \tag{S–7}$$

and $c_\kappa^{**} = \frac{c_\kappa^2 \Gamma(\delta_\kappa)^2}{(1-\delta_\kappa)^2} > 0$.

3 Proof of Lemma 8

Consider

$$f_y(\omega) - f_y(0) = \frac{(f_x(\omega) - f_x(0))(1-\beta)^2}{(1-\beta)^2 \left|1 - \beta + \beta\left(1 - \kappa\left(e^{-i\omega}\right)\right)\right|^2}$$
$$- f_x(0) \frac{2\beta(1-\beta)\,\mathrm{Re}\left[1 - \kappa\left(e^{-i\omega}\right)\right] + \beta^2 \left|1 - \kappa\left(e^{-i\omega}\right)\right|^2}{(1-\beta)^2 \left|1 - \beta + \beta\left(1 - \kappa\left(e^{-i\omega}\right)\right)\right|^2},$$

since

$$\left|1-\beta+\beta\left(1-\kappa\left(e^{-i\omega}\right)\right)\right|^{2}=(1-\beta)^{2}-2\beta\left(1-\beta\right)\mathrm{Re}\left[\kappa\left(e^{-i\omega}\right)-1\right]-\beta^{2}\left|\kappa\left(e^{-i\omega}\right)-1\right|^{2}.$$

Under Assumption B, $|f_x'(0)|<\infty$ (see Stock, 1994), so $f_x(\omega)-f_x(0)=O(\omega)$. Now if $\delta_\kappa>0$, under constant learning, $\kappa_j\sim c_\kappa j^{\delta_\kappa-2}$ for some $c_\kappa\neq 0$. Lemma 6 implies that there exist $c_\kappa^*\neq 0$, with $c_\kappa c_\kappa^*>0$, and $c_\kappa^{**}>0$ such that

$$\begin{aligned} f_y(\omega)-f_y(0) &\underset{\omega\to 0^+}{\sim} \frac{-2\beta\left(1-\beta\right)f_x(0)\,c_\kappa^*\omega^{1-\delta_\kappa}+\beta^2 f_x(0)\,c_\kappa^{**}\omega^{2(1-\delta_\kappa)}}{(1-\beta)^4} \qquad \text{(S–8)}\\ &\underset{\omega\to 0^+}{\sim} -\frac{2\beta f_x(0)\,c_\kappa^*}{(1-\beta)^3}\omega^{1-\delta_\kappa}. \end{aligned}$$

We first note that by definition of the population spectrum

$$f_y(\omega)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_k e^{-i\omega k}=\frac{1}{2\pi}\left\{\gamma_0+2\sum_{k=1}^{\infty}\gamma_k\cos\omega k\right\},$$

where $\gamma_k=\mathsf{Cov}\left(y_t,y_{t-k}\right)$, is symmetric since y_t is stationary. We assume for now that γ_k is of bounded variation, then, for $\omega\neq 0$, the series $\sum_{k=1}^{n}\gamma_k\cos\omega k$ converges uniformly as $n\to\infty$ (see Zygmund, 1935, Section. 1.23). It follows that the derivative of f_y satisfies:

$$f_y'(\omega)=-\frac{1}{\pi}\sum_{k=1}^{\infty}k\gamma_k\sin k\omega. \qquad \text{(S–9)}$$

We now use Theorem III-11 of Yong (1974) who works under the assumption that $\{a_k\}_{k\in\mathbb{N}}$ is a sequence of positive numbers that is quasi-monotonically convergent to zero (i.e. $a_k\to 0$ and there exist $M\geq 0$, such that $a_{k+1}\leq a_k\left(1+\frac{M}{k}\right)$ for all $k\geq k_0(M)$) and that $\{a_k\}$ is also of bounded variation, i.e. $\sum_{k=1}^{\infty}|\Delta a_k|<\infty$. The theorem states that for $a\in(0,1)$, $a_k\sim k^{-a}S^*(k)$ as $k\to\infty$, with S^* slowly varying, if and only if

$$\sum_{k=1}^{\infty}a_k\sin k\omega\sim\frac{\pi}{2\Gamma(a)\sin\frac{\pi a}{2}}\omega^{a-1}S^*\left(\frac{1}{\omega}\right)\ \text{ as }\omega\to 0^+.$$

We apply this theorem to (S–9), using expression (S–8) that $f_y'(\omega)\underset{\omega\to 0^+}{\sim}-\frac{2\beta f_x(0)c_\kappa^*}{(1-\beta)^3}\omega^{-\delta_\kappa}$

$$-\frac{1}{\pi}\sum_{k=1}^{\infty}k\gamma_k\sin k\omega\underset{\omega\to 0^+}{\sim}-\frac{2\beta f_x(0)\,c_\kappa^*}{(1-\beta)^3}\omega^{-\delta_\kappa}.$$

We let $a=1-\delta_\kappa$ in the theorem of Yong above, defining

$$a_k=\frac{k\gamma_k}{\pi}\frac{(1-\beta)^3}{2\beta f_x(0)\,c_\kappa^*}.$$

This implies that $a_k \sim \frac{\pi}{2\Gamma(a)\sin\frac{\pi a}{2}} k^{-(1-\delta_\kappa)}$, with $\Gamma(1-\delta_\kappa)\sin\frac{\pi(1-\delta_\kappa)}{2} = \frac{\pi}{2\Gamma(\delta_\kappa)\sin\frac{\pi\delta_\kappa}{2}}$, i.e.

$$\gamma_k \sim \frac{2\pi\beta f_x(0)\, c_\kappa^* \Gamma(\delta_\kappa)\sin\frac{\pi\delta_\kappa}{2}}{(1-\beta)^3} k^{-(2-\delta_\kappa)}. \tag{S–10}$$

To apply Theorem III-11 of Yong (1974), we check that a_k thus defined is a quasi-monotonic sequence with bounded variation. The first holds since $a_{k+1}/a_k \sim (1+1/k)^{-(1-\delta_\kappa)} < 1$, so choose M such that $a_{k+1}/a_k < 1$ for $k > M$. Also $k\gamma_k$ is clearly of bounded variation since it is asymptotically positive and

$$\Delta(k\gamma_k) = O\left(k^{-(2-\delta_\kappa)}\right)$$

is summable. Finally, we check the uniform convergence condition in Zygmund (1935): $|\Delta\gamma_k| = O\left(k^{-(3-\delta_\kappa)}\right)$ so γ_k is of bounded variation.

References

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