

Supplementary Appendix to:
Probabilistic Forecasting of Bubbles and Crashes.

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1 Technical Proofs

We collect here the proofs that R_t and R_t^* can be neglected in the expressions where they appear in the proofs above.

1.1 Case of R_t

We show here that the residual $R_{[\kappa_T]}$ appearing in S_{1T} , expression (25) is asymptotically negligible in probability with respect to the other terms.

Writing $R_t = R_{1t} - 2R_{2t}$ with

$$\begin{aligned} R_{1t} &= \sum_{k=1}^t \left[\sum_{i=k+1}^t \exp \left(\frac{\phi}{T^\alpha} (k-i) - \frac{\lambda}{T^{\alpha/2}} (U_k - U_i) \right) \eta_i \right]^2 \\ R_{2t} &= \left(\sum_{i=1}^t \exp \left(-\frac{\phi}{T^\alpha} i - \frac{\lambda}{T^{\alpha/2}} U_i \right) \eta_i \right) \times \sum_{k=1}^t \sum_{i=k+1}^t \exp \left(\frac{\phi}{T^\alpha} (2k-i) + \frac{\lambda}{T^{\alpha/2}} (2U_k - U_i) \right) \eta_i \\ &\equiv \left(\sum_{i=1}^t \exp \left(-\frac{\phi}{T^\alpha} i - \frac{\lambda}{T^{\alpha/2}} U_i \right) \eta_i \right) \times \bar{R}_{2t}. \end{aligned}$$

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First,

$$T^{-2\alpha} \psi_{[T^{1-\alpha}]}^{-1} \varphi_{[T^{1-\alpha}]}^{-2} R_{1[\kappa_T]} = \psi_{[T^{1-\alpha}]}^{-1} \int_0^{[T^{1-\alpha}]} \left(\sigma_\eta \varphi_{[T^{1-\alpha}]}^{-1} \int_r^{[T^{1-\alpha}]} e^{\phi(r-s) + \lambda(W_r - W_s)} dB_s \right)^2 dr + o_p(1)$$

uniformly, and since we also have

$$\begin{aligned} \mathbb{E} \left(\int_r^{[T^{1-\alpha}]} e^{\phi(r-s) - \lambda(W_r - W_s)} dB_s \right)^2 &= \int_r^{[T^{1-\alpha}]} e^{2(\phi + \lambda^2)(r-s)} ds \\ &= \frac{1 - e^{-2(\phi + \lambda^2)([T^{1-\alpha}] - r)}}{2(\phi + \lambda^2)} = O(1) \end{aligned}$$

uniformly in $r \leq T^{1-\alpha}$, we obtain

$$T^{-2\alpha} \psi_{[T^{1-\alpha}]}^{-1} \varphi_{[T^{1-\alpha}]}^{-2} R_{1[\kappa_T]} = O_p(1) \times O \left(\psi_{[T^{1-\alpha}]}^{-1} \int_0^{[T^{1-\alpha}]} dr \right) = O_p \left([T^{1-\alpha}] \psi_{[T^{1-\alpha}]}^{-1} \right).$$

Next, we have $T^{-\alpha} \psi_{[T^{1-\alpha}]}^{-1} \bar{R}_{2[\kappa_T]} = O_p \left([T^{1-\alpha}] \psi_{[T^{1-\alpha}]}^{-1} \right)$, so it follows that

$$T^{-2\alpha} \psi_{[T^{1-\alpha}]}^{-1} \varphi_{[T^{1-\alpha}]}^{-2} R_{2[\kappa_T]} = O_p \left(T^{1-3/2\alpha} \psi_{[T^{1-\alpha}]}^{-1} \varphi_{[T^{1-\alpha}]}^{-1} \right),$$

hence the result concerning $R_{[\kappa_T]}$.

1.2 Case of R_t^*

We check that the term in R_t^* could indeed be neglected in expression (32). It is enough to notice that R_t^* can be expressed as

$$\begin{aligned} R_t^* &= T^{\alpha/2} \sum_{t=0}^{T-1} \exp \left(\frac{2\phi}{T^\alpha} t + \frac{2\lambda}{T^{\alpha/2}} U_t \right) \left[\sum_{i=1}^t \exp \left(-\frac{\phi}{T^\alpha} i - \frac{\lambda}{T^{\alpha/2}} U_i \right) \eta_i \right]^2 \Delta U_{t+1}^\rho \\ &\quad - T^{\alpha/2} \left(\sum_{k=1}^{T-1} \exp \left(\frac{2\phi}{T^\alpha} k + \frac{2\lambda}{T^{\alpha/2}} U_k \right) \Delta U_{k+1}^\rho \right) \left[\sum_{i=1}^T \exp \left(-\frac{\phi}{T^\alpha} i - \frac{\lambda}{T^{\alpha/2}} U_i \right) \eta_i \right]^2 \\ &\equiv R_{1t}^* - 2R_{2t}^*, \end{aligned}$$

with $R_{1t}^* \equiv T^{\alpha/2} \sum_{k=1}^t \left[\sum_{i=k+1}^t \exp \left(\frac{\phi}{T^\alpha} (k-i) - \frac{\lambda}{T^{\alpha/2}} (U_k - U_i) \right) \eta_i \right]^2 \Delta U_{k+1}^\rho$, and

$$R_{2t}^* \equiv \sum_{i=1}^t \exp \left(-\frac{\phi}{T^\alpha} i - \frac{\lambda}{T^{\alpha/2}} U_i \right) \eta_i T^{\alpha/2} \sum_{k=1}^t \sum_{i=k+1}^t \exp \left(\frac{\phi}{T^\alpha} (2k-i) + \frac{\lambda}{T^{\alpha/2}} (2U_k - U_i) \right) \eta_i \Delta U_{k+1}^\rho.$$

We write the latter as $R_{2t}^* \equiv \sum_{i=1}^t \exp\left(-\frac{\phi}{T^\alpha}i - \frac{\lambda}{T^{\alpha/2}}U_i\right) \eta_i \times \bar{R}_{2t}^*$, in order to follow the same lines as the proof for R_t in the previous subsection. Notice that to do so, we let $T^{\alpha/2}$ appear explicitly in the definitions of R_t^* , R_{1t}^* and R_{2t}^* above, so that $V[T^{\alpha/2}\Delta U_{k+1}^\rho] \rightarrow \lambda^2 \neq 0$ as $T \rightarrow \infty$. The result follows.

2 Proof of Corollary 3

We write

$$\hat{\rho} - \mathbb{E}_{\mathbb{H}_0}[\rho_t] = (\hat{\rho} - \mathbb{E}_{\mathbb{H}_1}[\rho_t]) + (\mathbb{E}_{\mathbb{H}_1}[\rho_t] - \mathbb{E}_{\mathbb{H}_0}[\rho_t]),$$

and consider the two elements of the sum in turn.

The expectations under the null and alternative hypotheses are local to each other:

$$\mathbb{E}_{\mathbb{H}_1}[\rho_t] - \mathbb{E}_{\mathbb{H}_0}[\rho_t] = \frac{\phi_1 - \phi_0 + \frac{1}{2}(\lambda_1^2 - \lambda_0^2)}{T^\alpha} + o(T^{-\alpha}),$$

hence $T^{\frac{1+\alpha}{2}}(\mathbb{E}_{\mathbb{H}_1}[\rho_t] - \mathbb{E}_{\mathbb{H}_0}[\rho_t])$ diverges but $T^\alpha(\mathbb{E}_{\mathbb{H}_1}[\rho_t] - \mathbb{E}_{\mathbb{H}_0}[\rho_t]) = \phi_1 - \phi_0 + (\lambda_1^2 - \lambda_0^2)/2 + o(1)$ does not. Also, under the alternative, $T^{\frac{1+\alpha}{2}}(\hat{\rho} - \mathbb{E}_{\mathbb{H}_1}[\rho_t])$ diverges only if $\phi_1 + \lambda_1^2 \geq 0$ but $T^\alpha(\hat{\rho} - \mathbb{E}_{\mathbb{H}_1}[\rho_t])$ does not diverge.

Finally, if both $T^{\frac{1+\alpha}{2}}(\hat{\rho} - \mathbb{E}_{\mathbb{H}_1}[\rho_t])$ and $T^{\frac{1+\alpha}{2}}(\mathbb{E}_{\mathbb{H}_1}[\rho_t] - \mathbb{E}_{\mathbb{H}_0}[\rho_t])$ diverge. Their sum is $O_p\left(T^{\frac{1-\alpha}{2}}\right)$ so they do not cancel each other.

To conclude, $\delta_{\theta_0, T}$ diverges under \mathbb{H}_1 only if $\phi_0 + \lambda_0^2 < 0$, irrespective of (ϕ_1, λ_1) .