Supplementary Appendix to:
Probabilistic Forecasting of Bubbles and Crashes.

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1 Technical Proofs

We collect here the proofs that R_t and R_t^* can be neglected in the expressions where they appear in the proofs above.

1.1 Case of R_t

We show here that the residual $R_{\lfloor \kappa_T \rfloor}$ appearing in S_{1T}, expression (25) is asymptotically negligible in probability with respect to the other terms.

Writing $R_t = R_{1t} - 2R_{2t}$ with

$$R_{1t} = \sum_{k=1}^{t} \left[\sum_{i=k+1}^{t} \exp\left(\frac{\phi}{T^\alpha} (k-i) - \frac{\lambda}{T^{\alpha/2}} (U_k - U_i) \right) \eta_i \right]^2$$

$$R_{2t} = \left(\sum_{i=1}^{t} \exp\left(-\frac{\phi}{T^\alpha} i - \frac{\lambda}{T^{\alpha/2}} U_i \right) \eta_i \right) \times \sum_{k=1}^{t} \sum_{i=k+1}^{t} \exp\left(\frac{\phi}{T^\alpha} (2k-i) + \frac{\lambda}{T^{\alpha/2}} (2U_k - U_i) \right) \eta_i$$

$$\equiv \left(\sum_{i=1}^{t} \exp\left(-\frac{\phi}{T^\alpha} i - \frac{\lambda}{T^{\alpha/2}} U_i \right) \eta_i \right) \times \overline{R}_{2t}.$$

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First,

$$T^{-2\alpha}\ \psi_{\lfloor T^{1-\alpha}\rfloor}^{-1}\varphi_{\lfloor T^{1-\alpha}\rfloor}^{-2}R_{1\lfloor\kappa_T\rfloor}=\psi_{\lfloor T^{1-\alpha}\rfloor}^{-1}\int_0^{\lfloor T^{1-\alpha}\rfloor}\left(\sigma_\eta\varphi_{\lfloor T^{1-\alpha}\rfloor}^{-1}\int_r^{\lfloor T^{1-\alpha}\rfloor}e^{\phi(r-s)+\lambda(W_r-W_s)}dB_s\right)^2dr+o_p\left(1\right)$$

uniformly, and since we also have

$$\begin{aligned}\mathsf{E}\left(\int_r^{\lfloor T^{1-\alpha}\rfloor}e^{\phi(r-s)-\lambda(W_r-W_s)}dB_s\right)^2&=\int_r^{\lfloor T^{1-\alpha}\rfloor}e^{2\left(\phi+\lambda^2\right)(r-s)}ds\\&=\frac{1-e^{-2\left(\phi+\lambda^2\right)\left(\lfloor T^{1-\alpha}\rfloor-r\right)}}{2\left(\phi+\lambda^2\right)}=O\left(1\right)\end{aligned}$$

uniformly in $r\le T^{1-\alpha}$, we obtain

$$T^{-2\alpha}\psi_{\lfloor T^{1-\alpha}\rfloor}^{-1}\varphi_{\lfloor T^{1-\alpha}\rfloor}^{-2}R_{1\lfloor\kappa_T\rfloor}=O_p\left(1\right)\times O\left(\psi_{\lfloor T^{1-\alpha}\rfloor}^{-1}\int_0^{\lfloor T^{1-\alpha}\rfloor}dr\right)=O_p\left(\lfloor T^{1-\alpha}\rfloor\psi_{\lfloor T^{1-\alpha}\rfloor}^{-1}\right).$$

Next, we have $T^{-\alpha}\psi_{\lfloor T^{1-\alpha}\rfloor}^{-1}\overline{R}_{2\lfloor\kappa_T\rfloor}=O_p\left(\lfloor T^{1-\alpha}\rfloor\psi_{\lfloor T^{1-\alpha}\rfloor}^{-1}\right)$, so it follows that

$$T^{-2\alpha}\psi_{\lfloor T^{1-\alpha}\rfloor}^{-1}\varphi_{\lfloor T^{1-\alpha}\rfloor}^{-2}R_{2\lfloor\kappa_T\rfloor}=O_p\left(T^{1-3/2\alpha}\psi_{\lfloor T^{1-\alpha}\rfloor}^{-1}\varphi_{\lfloor T^{1-\alpha}\rfloor}^{-1}\right),$$

hence the result concerning $R_{\lfloor\kappa_T\rfloor}$.

1.2 Case of R_t^*

We check that the term in R_t^* could indeed be neglected in expression (32). It is enough to notice that R_t^* can be expressed as

$$\begin{aligned}R_t^*=&T^{\alpha/2}\sum_{t=0}^{T-1}\exp\left(\frac{2\phi}{T^\alpha}t+\frac{2\lambda}{T^{\alpha/2}}U_t\right)\left[\sum_{i=1}^{t}\exp\left(-\frac{\phi}{T^\alpha}i-\frac{\lambda}{T^{\alpha/2}}U_i\right)\eta_i\right]^2\Delta U_{t+1}^\rho\\&-T^{\alpha/2}\left(\sum_{k=1}^{T-1}\exp\left(\frac{2\phi}{T^\alpha}k+\frac{2\lambda}{T^{\alpha/2}}U_k\right)\Delta U_{k+1}^\rho\right)\left[\sum_{i=1}^{T}\exp\left(-\frac{\phi}{T^\alpha}i-\frac{\lambda}{T^{\alpha/2}}U_i\right)\eta_i\right]^2\\\equiv&R_{1t}^*-2R_{2t}^*,\end{aligned}$$

with $R_{1t}^*\equiv T^{\alpha/2}\sum_{k=1}^{t}\left[\sum_{i=k+1}^{t}\exp\left(\frac{\phi}{T^\alpha}\left(k-i\right)-\frac{\lambda}{T^{\alpha/2}}\left(U_k-U_i\right)\right)\eta_i\right]^2\Delta U_{k+1}^\rho$, and

$$R_{2t}^*\equiv\sum_{i=1}^{t}\exp\left(-\frac{\phi}{T^\alpha}i-\frac{\lambda}{T^{\alpha/2}}U_i\right)\eta_iT^{\alpha/2}\sum_{k=1}^{t}\sum_{i=k+1}^{t}\exp\left(\frac{\phi}{T^\alpha}\left(2k-i\right)+\frac{\lambda}{T^{\alpha/2}}\left(2U_k-U_i\right)\right)\eta_i\Delta U_{k+1}^\rho.$$

We write the latter as $R^*_{2t} \equiv \sum_{i=1}^{t} \exp\left(-\frac{\phi}{T^\alpha} i - \frac{\lambda}{T^{\alpha/2}} U_i\right) \eta_i \times \overline{R}^*_{2t}$, in order to follow the same lines as the proof for R_t in the previous subsection. Notice that to do so, we let $T^{\alpha/2}$ appear explicitly in the definitions of R^*_t, R^*_{1t} and R^*_{2t} above, so that $\mathsf{V}\left[T^{\alpha/2} \Delta U^{\rho}_{k+1}\right] \to \lambda^2 \neq 0$ as $T \to \infty$. The result follows.

2 Proof of Corollary 3

We write

$$\widehat{\rho} - \mathsf{E}_{\mathsf{H}_0}[\rho_t] = \left(\widehat{\rho} - \mathsf{E}_{\mathsf{H}_1}[\rho_t]\right) + \left(\mathsf{E}_{\mathsf{H}_1}[\rho_t] - \mathsf{E}_{\mathsf{H}_0}[\rho_t]\right),$$

and consider the two elements of the sum in turn.

The expectations under the null and alternative hypotheses are local to each other:

$$\mathsf{E}_{\mathsf{H}_1}[\rho_t] - \mathsf{E}_{\mathsf{H}_0}[\rho_t] = \frac{\phi_1 - \phi_0 + \frac{1}{2}\left(\lambda_1^2 - \lambda_0^2\right)}{T^\alpha} + o\left(T^{-\alpha}\right),$$

hence $T^{\frac{1+\alpha}{2}} \left(\mathsf{E}_{\mathsf{H}_1}[\rho_t] - \mathsf{E}_{\mathsf{H}_0}[\rho_t]\right)$ diverges but $T^\alpha \left(\mathsf{E}_{\mathsf{H}_1}[\rho_t] - \mathsf{E}_{\mathsf{H}_0}[\rho_t]\right) = \phi_1 - \phi_0 + \left(\lambda_1^2 - \lambda_0^2\right)/2 + o(1)$ does not. Also, under the alternative, $T^{\frac{1+\alpha}{2}} \left(\widehat{\rho} - \mathsf{E}_{\mathsf{H}_1}[\rho_t]\right)$ diverges only if $\phi_1 + \lambda_1^2 \geq 0$ but $T^\alpha \left(\widehat{\rho} - \mathsf{E}_{\mathsf{H}_1}[\rho_t]\right)$ does not diverge.

Finally, if both $T^{\frac{1+\alpha}{2}} \left(\widehat{\rho} - \mathsf{E}_{\mathsf{H}_1}[\rho_t]\right)$ and $T^{\frac{1+\alpha}{2}} \left(\mathsf{E}_{\mathsf{H}_1}[\rho_t] - \mathsf{E}_{\mathsf{H}_0}[\rho_t]\right)$ diverge. Their sum is $O_p\left(T^{\frac{1-\alpha}{2}}\right)$ so they do not cancel each other.

To conclude, $\delta_{\theta_0,T}$ diverges under H_1 only if $\phi_0 + \lambda_0^2 < 0$, irrespective of (ϕ_1, λ_1).