Generating Univariate Fractional Integration within a Large VAR(1), Supplementary Appendix

Guillaume Chevillon*
ESSEC Business School & CREST

Alain Hecq†
Maastricht University

Sébastien Laurent‡
Aix-Marseille University

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*Department of Information Systems, Decision Sciences and Statistics, ESSEC Business School, Avenue Bernard Hirsch, 95021 Cergy-Pontoise, France. Email: chevillon@essec.edu
†Department of Quantitative Economics, Maastricht University, The Netherlands.
Email: a.hecq@maastrichtuniversity.nl
‡Aix-Marseille University (Aix-Marseille School of Economics), CNRS & EHESS, Aix-Marseille Graduate School of Management-IAE, France. Email: sebastien.laurent@univ-amu.fr

1 Examples relating to Remark 2

1.1 Example of ARFIMA$(p, \delta, 0)$

Consider $\mathbf{y}_t$ such that $\mathbf{F}_n(L)\,\mathbf{y}_t = \mathbf{x}_t$, with

$$\mathbf{F}_n(L) = \mathbf{I}_n - \begin{bmatrix} a & 0 & \mathbf{0}'_{n-2} \\ b & c & \mathbf{e}' \\ \mathbf{d} & \mathbf{0}_{n-2} & \mathbf{0}_{(n-2)\times(n-2)} \end{bmatrix} L,$$

where a, b, c are scalar, $\mathbf{e}$ and $\mathbf{d}$ denote vectors of dimension $(n-2)$, and $\mathbf{0}_{n-2}$ is a vector of zeros of dimension $(n-2)$. The FER is

$$(1-aL)(1-cL)\,\mathbf{y}_t = \begin{bmatrix} 1-cL & 0 & \mathbf{0}_{1\times(n-2)} \\ (b-\mathbf{e}'\mathbf{d}L)\,L & 1-cL & (1-aL)\,L\mathbf{e}' \\ (1-cL)\,L\mathbf{d} & \mathbf{0}_{(n-2)\times 1} & (1-aL)(1-cL)\,\mathbf{I}_{n-2} \end{bmatrix} \mathbf{x}_t$$

so for $j=1$, $(1-aL)(1-cL)\,y_{1t} = (1-cL)\,x_{1t}$, i.e. cancelling the common factor $(1-aL)\,y_{1t} = x_{1t}$. As $n \to \infty$, under the assumptions of Theorem 1 with $j=1$,

$$(1-aL)\,y_{1t} \xrightarrow{L} \Delta^{-\delta}\epsilon_{1t}.$$

This example generalizes to

$$\mathbf{F}_n(L) = \mathbf{I}_n - \begin{bmatrix} a & \mathbf{0}_{1\times(n-1)} \\ \mathbf{b}_{(n-1)\times 1} & \mathbf{C}_{(n-1)\times(n-1)} \end{bmatrix} L$$

which yields an asymptotic ARFIMA$(1, \delta, 0)$ for y_{1t}.

1.2 Example of ARFIMA(p, δ, q)

Let $\mathbf{F}_n(L)\,\mathbf{y}_t = \mathbf{x}_t$ and assume

$$\mathbf{F}_n(L) = \begin{bmatrix} h(L) & 1 & \mathbf{0}'_{n-2} \\ -\alpha(L) + h(L)\,\beta(L) & \beta(L) & \mathbf{0}'_{n-2} \\ \mathbf{0}_{n-2} & \mathbf{0}_{n-2} & \mathbf{I}_{n-2} \end{bmatrix},$$

where $\mathbf{0}_{n-2}$ denotes a vector of zeros of dimension $(n-2)$, $\alpha(L)$ and $\beta(L)$ are invertible scalar polynomials of fixed orders p and q with no common roots, $h(L)$ denotes a scalar polynomial. Using the FER for $\mathbf{y}_t$, we obtain

$$\alpha(L)\,\mathbf{y}_t = \begin{bmatrix} \beta(L) & 1 & \mathbf{0}'_{n-2} \\ -\alpha(L) + h(L)\,\beta(L) & h(L) & \mathbf{0}'_{n-2} \\ \mathbf{0}_{n-2} & \mathbf{0}_{n-2} & \alpha(L) \end{bmatrix} \mathbf{x}_t.$$

Under the conditions of Theorem 1, with $j=1$, and as $n \to \infty$, $x_{1t} \xrightarrow{L} \Delta^{-\delta}\epsilon_{1t}$ so $z_t = y_{1t} - \alpha(L)^{-1} x_{2t}$ tends to a process solution to

$$\alpha(L)\,z_t = \beta(L)\,\Delta^{-\delta}\epsilon_{1t},$$

i.e. an ARFIMA(p, δ, q).

2 A useful Lemma

In subsequent proofs, we use the following lemma.

Lemma L Under the assumptions of both Examples 1 and 2 and as $n\to\infty$, the coefficients of $\mathbf{T}_n^*$ satisfy for all k, $-n<k<n$, $t_k^{*(n-1)}-t_k^{*(n)}\sim\frac{1}{n}t_k^{*(n)}$ for all n such that $(n-1)/4\in\mathbb{N}$.

Proof. Recall that $t_k^{*(n)}=\mathrm{Re}\Big[\frac{1}{n}\sum_{\ell=0}^{n-1}g\left(\delta_n,e^{i\frac{2\pi\ell}{n}}\right)e^{-2i\pi\ell k/n}\Big]$, where $\mathrm{Re}\left[\cdot\right]$ denotes the real part, so

$$\begin{aligned}&t_k^{*(n-1)}-t_k^{*(n)}\\&=\frac{1}{n\left(n-1\right)}\mathrm{Re}\left[n\sum_{\ell=0}^{n-2}g\left(\delta_{n-1},e^{i\frac{2\pi\ell}{n-1}}\right)e^{-2i\pi\ell k/(n-1)}-\left(n-1\right)\sum_{\ell=0}^{n-1}g\left(\delta_n,e^{i\frac{2\pi\ell}{n}}\right)e^{-2i\pi\ell k/n}\right]\\&=\frac{1}{n-1}\mathrm{Re}\left[\sum_{\ell=0}^{n-2}\left[g\left(\delta_{n-1},e^{i\frac{2\pi\ell}{n-1}}\right)e^{-2i\pi\ell k/(n-1)}-g\left(\delta_n,e^{i\frac{2\pi\ell}{n}}\right)e^{-2i\pi\ell k/n}\right]\right.\\&\left.+\frac{1}{n\left(n-1\right)}\sum_{\ell=0}^{n-1}g\left(\delta_n,e^{i\frac{2\pi\ell}{n}}\right)e^{-2i\pi\ell k/n}-\frac{1}{n}g\left(\delta_n,e^{i\frac{2\pi(n-1)}{n}}\right)e^{-2i\pi(n-1)k/n}\right].\end{aligned}$$

Hence, as $n^{-1}=\left(n-1\right)^{-1}\left(1-n^{-1}\right)$,

$$\begin{aligned}&t_k^{*(n-1)}-t_k^{*(n)}\\&=\frac{1}{n-1}\mathrm{Re}\left[\sum_{\ell=0}^{n-2}\left[g\left(\delta_{n-1},e^{i\frac{2\pi\ell}{n-1}}\right)-g\left(\delta_n,e^{i\frac{2\pi\ell}{n}}\right)e^{2i\pi\ell k\frac{1}{n(n-1)}}\right]e^{-2i\pi\ell k/(n-1)}\right]\\&+\frac{1}{n-1}t_k^{*(n)}-\frac{1}{n}\mathrm{Re}\left[g\left(\delta_n,e^{i\frac{2\pi(n-1)}{n}}\right)e^{-2i\pi(n-1)k/n}\right].\end{aligned}$$

Recall the definition $g\left(\delta,e^{i\omega}\right)=1_{\{0\leq u<\pi\delta\}}+1_{\left\{\pi\left(\frac{3}{2}-\delta\right)<u\leq\frac{3\pi}{2}\right\}}$ for $\omega=u \bmod 2\pi\geq 0$, so g is real, and

$$g\left(\delta_n,e^{i\frac{2\pi\ell}{n}}\right)=1_{\left\{0\leq\frac{2\pi\ell}{n}<\pi\delta_n\right\}}+1_{\left\{\pi\left(\frac{3}{2}-\delta_n\right)<\frac{2\pi\ell}{n}\leq\frac{3\pi}{2}\right\}}.$$

We assumed $\delta_n=\delta+o\left(n^{-2}\right)$ with $\left(\delta_n\right)$ a nondecreasing sequence. Hence, for all δ and n large enough, for all $\ell<n$

$$\frac{2\ell}{n-1}<\delta_{n-1}\Leftrightarrow\frac{2\ell}{n}<\delta_n\Leftrightarrow\frac{2\ell}{n}<\delta,$$

i.e. $1_{\left\{0\leq\frac{2\pi\ell}{n-1}<\pi\delta_{n-1}\right\}}=1_{\left\{0\leq\frac{2\pi\ell}{n}<\pi\delta_n\right\}}=1_{\left\{0\leq\frac{2\pi\ell}{n}<\pi\delta\right\}}$. Also, for n large enough,

$$\left(\frac{3}{2}-\delta_{n-1}\right)<\frac{2\ell}{n-1}\Leftrightarrow\left(\frac{3}{2}-\delta_n\right)<\frac{2\ell}{n}\Leftrightarrow\left(\frac{3}{2}-\delta\right)<\frac{2\ell}{n}$$

so $1_{\left\{\pi\left(\frac{3}{2}-\delta_{n-1}\right)<\frac{2\pi\ell}{n-1}\right\}}=1_{\left\{\pi\left(\frac{3}{2}-\delta_n\right)<\frac{2\pi\ell}{n}\right\}}=1_{\left\{\pi\left(\frac{3}{2}-\delta\right)<\frac{2\pi\ell}{n}\right\}}$. Finally, for all n, let $(m,s)\in\mathbb{N}\times\{0,1,2,3\}$ such that $n=4m+s$. Clearly $\frac{2\pi\ell}{n-1}\leq\frac{3\pi}{2}\Rightarrow\frac{2\pi\ell}{n}\leq\frac{3\pi}{2}$, now, for the converse,

$$\frac{2\pi\ell}{n}<\frac{3\pi}{2}\Leftrightarrow\ell<\frac{3n}{4}=3m+\frac{3}{4}s,$$

hence, since ℓ is an integer $\ell\leq\frac{3n}{4}\Rightarrow\ell\leq 3m+s_s^*$, where $s_0^*=s_1^*=0$ and $s_s^*=s-1$ for $s\in\{2,3\}$. Therefore

$$\frac{\ell}{n-1}\leq\frac{3m+s_s^*}{4m+s-1}=\frac{3}{4}+\frac{\left(4s_s^*-3s+3\right)/4}{4m+s-1},$$

where $4s_s^* - 3s + 3 \leq 0$ for $s = 1$. Hence, for $(n-1)/4 \in \mathbb{N}$,

$$1_{\left\{\frac{2\pi\ell}{n-1} \leq \frac{3\pi}{2}\right\}} = 1_{\left\{\frac{2\pi\ell}{n} \leq \frac{3\pi}{2}\right\}}.$$

Therefore, there exists N such that if $n > N$ and $(n-1)/4 \in \mathbb{N}$, $g\left(\delta_{n-1}, e^{i\frac{2\pi\ell}{n-1}}\right) = g\left(\delta_n, e^{i\frac{2\pi\ell}{n}}\right) = g\left(\delta, e^{i\frac{2\pi\ell}{n}}\right)$ so that

$$t_k^{*(n-1)} - t_k^{*(n)} = \frac{1}{n-1} \operatorname{Re}\left[\sum_{\ell=0}^{n-2} g\left(\delta, e^{i\frac{2\pi\ell}{n}}\right)\left[e^{-2i\pi\ell k/n} - e^{-2i\pi\ell k/(n-1)}\right]\right] + \frac{1}{n-1} t_k^{*(n)} - \frac{1}{n} g\left(\delta_n, e^{i\frac{2\pi(n-1)}{n}}\right) \cos\frac{2\pi(n-1)k}{n}.$$

The definition of $g(\cdot,\cdot)$ implies that as $n \to \infty$, $g\left(\delta_n, e^{i\frac{2\pi(n-1)}{n}}\right) \to \lim_{\substack{u\to 2\pi\\ u<2\pi}} g\left(\delta, e^{iu}\right)$ which is zero, and in particular as $g(d,u)$ is identically zero in a neighborhood $u \in [2\pi - \epsilon, 2\pi)$ for d in a neighborhood of δ ($\delta > 0$). Hence, there exists $M > 0$ such that for $n > M$, $g\left(\delta_n, e^{i\frac{2\pi(n-1)}{n}}\right) = 0$. Now, if $n > \max(M, N)$ and $(n-1)/4 \in \mathbb{N}$,

$$t_k^{*(n-1)} - t_k^{*(n)} = \frac{1}{n-1} \operatorname{Re}\left[\sum_{\ell=0}^{n-2} g\left(\delta, e^{i\frac{2\pi\ell}{n}}\right)\left[e^{-2i\pi\ell k/n} - e^{-2i\pi\ell k/(n-1)}\right]\right] + \frac{1}{n-1} t_{-k}^{*(n)}$$

with $(n-1)^{-1} \operatorname{Re}\left[\sum_{\ell=0}^{n-2} g\left(\delta, e^{i\frac{2\pi\ell}{n}}\right)\left[e^{-2i\pi\ell k/n} - e^{-2i\pi\ell k/(n-1)}\right]\right] \sim t_k^{*(n)} + t_k^{*(n)}/(n-1) - t_k^{*(n-1)}$. Hence

$$t_k^{*(n-1)} - t_k^{*(n)} \sim \frac{1}{n} t_k^{*(n)} \tag{S–1}$$

for n such that $(n-1)/4 \in \mathbb{N}$. □

3 Proofs relative to Example 1

We collect here the proofs related to Example 1 that shows that Assumptions T and P are satisfied for $\mathbf{A}_n = \mathbf{T}_n^* + \eta_n \mathbf{D}_n$, where $\mathbf{T}_n^*$, η_n and $\mathbf{D}_n$ are specified as in Section 2.

3.1 Proof of the validity of Assumption T for the matrix $\mathbf{T}_n$

Assumptions T(i), and T($iii.a$) follow from the definitions of g and δ_n. To prove that Assumption T(ii) holds, we need to show that $\mathbf{T}_{d,n}$ belongs to the Wiener class for all $d \in (0,1)$. This follows from the fact that the derivative $\frac{\partial}{\partial\omega} g\left(d, e^{i\omega}\right)$ is continuous at $\omega = 0$ for all $d > 0$. Hence, the Fourier series of $g\left(d, e^{i\omega}\right)$ is absolutely summable at $\omega = 0$ (see Whittaker, 1930-31), i.e., $\lim_{n\to\infty} \sum_{k=-(n-1)}^{n-1} |t_{d,k}| < \infty$. Hence Assumption T($ii$) holds. Now, for Assumption T($iii.b$), notice that for $\delta_n \leq \delta$,

$$t_{\delta,k}^{(n)} - t_k^{(n)} = \frac{1}{n} \sum_{\ell=\lceil n\delta_n/2\rceil}^{\lceil n\delta/2\rceil - 1} e^{-2i\pi k\ell/n} + \frac{1}{n} \sum_{\ell=\left\lfloor \frac{3n}{4} - \frac{n\delta}{2}\right\rfloor + 1}^{\left\lfloor \frac{3n}{4} - \frac{n\delta_n}{2}\right\rfloor} e^{-2i\pi k\ell/n}, \tag{S–2}$$

where the first sum becomes identically zero when $\lceil n\delta_n/2\rceil - 1 = \lceil n\delta/2\rceil - 1$, i.e. $\lceil n\delta/2\rceil - 1 < n\delta_n/2$. It is the case when $\frac{\lceil n\delta/2\rceil - 1}{n/2} < \delta - (\delta - \delta_n)$, i.e. $\delta - \delta_n < \frac{\lceil n\delta/2\rceil - n\delta/2 - 1}{n/2} = O\left(n^{-1}\right)$. Since $\delta - \delta_n = o\left(n^{-2}\right)$, there exists $N > 0$ such that the latter expression holds for $n > N$. The second term in Expression (S–2) is itself identically zero when $\left\lfloor \frac{3n}{4} - \frac{n\delta_n}{2}\right\rfloor = \left\lfloor \frac{3n}{4} - \frac{n\delta}{2}\right\rfloor$, i.e. for $\delta = 1/2$, $\left\lfloor \frac{n}{2} + \frac{n(\delta - \delta_n)}{2}\right\rfloor = \left\lfloor \frac{n}{2}\right\rfloor$ which holds for $\frac{n(\delta-\delta_n)}{2} < 1/2$ since $\frac{n}{2} - \left\lfloor \frac{n}{2}\right\rfloor \in \left\{0, \frac{1}{2}\right\}$. It suffices that $n(\delta - \delta_n) < 1$, which holds for n large enough. Therefore $n^2\left(t_{\delta,k}^{(n)} - t_k^{(n)}\right)$ also becomes identically zero for n large enough, i.e. T($iii.b$) holds.

3.2 Proof of the validity of Assumptions P(i)-(iv) for the matrix $\mathbf{A}_n = \mathbf{T}_n^* + \eta_n \mathbf{D}_n$

We let $j=1$ without loss of generality and start by showing that Assumption P(ii) holds. The elements of Σ_n are constant hence we set it to $\mathbf{I}_n$ without loss of generality.

Elements $\widetilde{\mathbf{B}_n(z)}_{1k}$, for $k=1,...,n$, of the first row of $\widetilde{\mathbf{B}_n(z)}$, satisfy $\widetilde{\mathbf{B}_n(z)}_{1k} = (-1)^{k+1}\det\left(\mathbf{CoB}_n(z)_{1k}\right)$, where $\mathbf{CoB}_n(z)_{\ell k}$ is the (ℓ,k) entry of the matrix of cofactors of $\mathbf{B}_n(z)$. We consider first $\mathbf{CoB}_n(z)_{11}$ which is

$$\mathbf{CoB}_n(z)_{11} = \begin{bmatrix} 1-t_0^{*(n)}z & -\left(t_1^{*(n)}+\eta_n\gamma_{23}^{(n)}\right)z & \cdots & -\left(t_{n-2}^{*(n)}+\eta_n\gamma_{2n}^{(n)}\right)z \\ -\left(t_1^{*(n)}+\eta_n\gamma_{32}^{(n)}\right)z & 1-t_0^{*(n)}z & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\left(t_1^{*(n)}+\eta_n\gamma_{(n-1)n}^{(n)}\right)z \\ -\left(t_{n-2}^{*(n)}+\eta_n\gamma_{n2}^{(n)}\right)z & \cdots & -\left(t_1^{*(n)}+\eta_n\gamma_{n(n-1)}^{(n)}\right)z & 1-t_0^{(*n)}z \end{bmatrix},$$

where $\gamma_{\ell k}$ denotes the (ℓ,k) entry of $\mathbf{D}_n$. Denoting respectively by $\mathbf{T}_n^{*(1)}$ and $\mathbf{D}_n^{(1)}$ the submatrices of $\mathbf{T}_n^*$ and $\mathbf{D}_n$ of dimension $n-1$ obtained by removing their first row and first column, $\mathbf{CoB}_n(z)_{11}$ can be written in a matrix form as

$$\mathbf{CoB}_n(z)_{11} = \mathbf{B}_{n-1}(z) - \left(\mathbf{T}_{n-1}^* - \mathbf{T}_n^{*(1)} + \eta_{n-1}\mathbf{D}_{n-1} - \eta_n\mathbf{D}_n^{(1)}\right)z. \tag{S–3}$$

Now Lemma L implies that for $(n-1)/4 \in \mathbb{N}$, $\mathbf{CoB}_n(z)_{11} = \mathbf{I}_{n-1} - z\mathbf{T}_{n-1}^* + \mathbf{O}\left(\frac{z}{n}\mathbf{T}_{n-1}^*\right)$, hence $\det\mathbf{CoB}_n(z)_{11} \sim \det\left(\mathbf{I}_{n-1} - z\mathbf{T}_{n-1}^*\right) = \det\mathbf{B}_{n-1}(z)$. This constitutes the first part of the proof.

We now turn to P(i). We first consider $\widetilde{\mathbf{B}_n(z)}_{1k}$, $\forall k \neq 1$, for $z<1$. By symmetry of the system, we can in fact focus the proof on $\widetilde{\mathbf{B}_n(z)}_{12}$. Ignoring $\eta_n\mathbf{D}_n$ which is of lower order, as $n\to\infty$:

$$\widetilde{\mathbf{B}_n(z)}_{12} \sim -\det\left(\begin{bmatrix} -t_1^{*(n)}z & -t_1^{*(n)}z & \cdots & -t_{n-2}^{*(n)}z \\ -t_2^{*(n)}z & 1-t_0^{*(n)}z & \ddots & \vdots \\ \vdots & \ddots & \ddots & -t_1^{*(n)}z \\ -t_{n-1}^{*(n)}z & -t_{n-3}^{*(n)}z & \cdots & 1-t_0^{*(n)}z \end{bmatrix}\right).$$

The key feature that is shared by all the $\widetilde{\mathbf{B}_n(z)}_{1k}$, for $k\neq 1$, is that one of their columns (here the first) contains no element from the diagonal of $\mathbf{B}_n(L)$ (where a 1 appears). Hence

$$\widetilde{\mathbf{B}_n(z)}_{12} \sim -\left(\max_{0<k<n}\left|t_k^{*(n)}\right|z\right)\det\left(\begin{bmatrix} \frac{-t_1^{*(n)}}{\max_{0<k<n}\left|t_k^{*(n)}\right|} & -t_2^{*(n)}z & \cdots & -t_{n-2}^{*(n)}z \\ \frac{-t_1^{*(n)}}{\max_{0<k<n}\left|t_k^{*(n)}\right|} & 1-t_0^{*(n)}z & \ddots & \vdots \\ \vdots & \vdots & \ddots & -t_1^{*(n)}z \\ \frac{-t_{n-1}^{*(n)}}{\max_{0<k<n}\left|t_k^{*(n)}\right|} & -t_{n-3}^{*(n)}z & \cdots & 1-t_0^{*(n)}z \end{bmatrix}\right).$$

Without loss of generality, we assume for instance that $\max_{0<k<n}\left|t_k^{*(n)}\right| = \left|t_1^{*(n)}\right|$. Lemma L$(i)$ shows that for $k\neq 0$, $t_k^{*(n)} = O\left(n^{-1}\right)$, hence,

$$\widetilde{\mathbf{B}_n(z)}_{12} = O\left(\frac{z}{n}\right)\det\left(\begin{bmatrix} -1 & \mathbf{O}\left(n^{-1}\right) \\ \mathbf{O}(1) & \mathbf{B}_{n-2}(z) \end{bmatrix}\right).$$

Using the formula for the determinant of partitioned matrices, the determinant on the right-hand side of the latter expression satisfies, as $n\to\infty$, $\det\left(\mathbf{B}_{n-2}(z) + \mathbf{O}\left(n^{-1}\right)\right) = O(\det(\mathbf{B}_{n-2}(z)))$, see Abadir and

Magnus (2005, result 12.30). Therefore

$$\widetilde{\mathbf{B}_n(z)}_{12} = O\left(n^{-1}\det\left(\mathbf{B}_{n-2}(z)\right)\right). \tag{S–4}$$

For any polynomial P, let deg P denote its degree. Now, we introduce the Hadamard polynomial product, which is defined for $P(z) = \sum_{k\geq 0} p_k z^k$ and $Q(z) = \sum_{k\geq 0} q_k z^k$ as $P \circ Q(z) = \sum_{k=0}^{\min(\deg P, \deg Q)} p_k q_k z^k$ and $P(z)^{\circ 2} = P(z) \circ P(z)$. Below, $[P(z)]_{z=1}$ refers to $P(z)$ evaluated at $z = 1$. Hence,

$$\mathsf{Var}\left[\sum_{k\neq j} \frac{\widetilde{\mathbf{B}_n}(z)_{jk}}{\det\left(\mathbf{B}_n(z)\right)}\epsilon_{kt}\right] = \sum_{k\neq j}\left|\frac{\widetilde{\mathbf{B}_n}(z)_{jk}}{\det\left(\mathbf{B}_n(z)\right)}\right|_{z=1}^{\circ 2}\sigma_{\epsilon_k}^2, \tag{S–5}$$

where Expression (S–4) implies

$$\sum_{k=2}^{n}\left|\frac{\widetilde{\mathbf{B}_n(z)}_{1k}}{\det\left(\mathbf{B}_n(z)\right)}\right|^{\circ 2} = \sum_{k=2}^{n} O\left(\left[n^{-1}\right]^2\left|\frac{\det\left(\mathbf{B}_{n-2}(z)\right)}{\det\left(\mathbf{B}_n(z)\right)}\right|^{\circ 2}\right) = O\left(n^{-1}\left|\frac{\det\left(\mathbf{B}_{n-2}(z)\right)}{\det\left(\mathbf{B}_n(z)\right)}\right|^{\circ 2}\right). \tag{S–6}$$

The circulant matrix $\mathbf{I}_n - z\mathbf{C}_n$ associated to $\mathbf{B}_n(z)$ has symbol $1 - g(\delta_n, \cdot)\, z$ since $\mathbf{D}_n$ is antisymmetric. Hence, as $n \to \infty$, under Assumption T (using the same argument as used in proving Theorem 1),

$$\frac{\det\left(\mathbf{B}_{n-1}(z)\right)}{\det\left(\mathbf{B}_n(z)\right)} \sim \frac{\det\left(\mathbf{I}_{n-1} - z\mathbf{C}_{n-1}\right)}{\det\left(\mathbf{I}_n - z\mathbf{C}_n\right)}. \tag{S–7}$$

The limit $(1-z)^{-1/2}$ is finite for $z < 1$ so $\left|\frac{\det(\mathbf{B}_{n-2}(z))}{\det(\mathbf{B}_n(z))}\right|^{\circ 2} = \left|\frac{\det(\mathbf{B}_{n-2}(z))\det(\mathbf{B}_{n-1}(z))}{\det(\mathbf{B}_{n-1}(z))\det(\mathbf{B}_n(z))}\right|^{\circ 2}$, i.e. as $n \to \infty$ $\left|\frac{\det(\mathbf{B}_{n-2}(z))}{\det(\mathbf{B}_n(z))}\right|^{\circ 2} \to \left|(1-z)^{-1}\right|^{\circ 2}$, where $\left|(1-z)^{-1}\right|^{\circ 2} = \sum_{k=0}^{\infty}|z|^k = (1-|z|)^{-1}$. Hence the expression on the right-hand side in Expression (S–6) is $O\left(n^{-1}\right)$ when $z \neq 1$.

Now for $z \to 1$, the truncated polynomial $\left[(1-z)^{-1}\right]^{+}$ evaluated at $z = 1$ takes the value t so $\sum_{k=2}^{n}\left|\frac{\widetilde{\mathbf{B}_n(z)}_{1k}}{\det(\mathbf{B}_n(z))}\right|_{z=1}^{\circ 2} = O\left(n^{-1}\right)$. Together with Expression (S–5), this shows that Assumption P(i) holds.

We now conclude by proving the validity of Assumption P(iv). The assumption follows from Assumption T for $\mathbf{T}_n$. By construction, $\mathbf{T}_n^*$ is real valued and bounded. By transitivity of asymptotic equivalence (see Gray, 2006, Theorem 2.1),

$$\frac{\det\left(\mathbf{I}_{n-1} - \mathbf{T}_{n-1}^* z\right)}{\det\left(\mathbf{I}_n - \mathbf{T}_n^* z\right)} \sim \frac{\det\left(\mathbf{I}_{n-1} - \mathbf{C}_{n-1} z\right)}{\det\left(\mathbf{I}_n - \mathbf{C}_n z\right)} \sim \frac{\det\left(\mathbf{I}_{n-1} - \mathbf{T}_{n-1} z\right)}{\det\left(\mathbf{I}_n - \mathbf{T}_n z\right)}.$$

Now the circulant matrix associated with $\eta_n\mathbf{D}_n$ has negligible asymptotic entries so $\mathbf{A}_n \sim \mathbf{C}_n$ and the result follows.

4 Proofs relative to Example 2

In this example, many of the results follow from Lemma L and the proofs of Example 1. Assumptions T(i) and T(ii), and T($iii.a$) hold as shown in Subsection 3.1. Now $\delta_n = \delta$ for all n so $t_{\delta,k}^{(n)} - t_k^{(n)} \equiv 0$ and Assumption T($iii.b$) holds as well. We now consider Assumption P(i)-(iv). The proof of P(ii) follows the lines of the proof provided in Example 1, so $\widetilde{\mathbf{B}_n(z)}_{11} \sim \det\left(\mathbf{B}_{n-1}(z)\right)$ as $n \to \infty$. As for Assumption P(i), if $\sigma_{n,k}^2 = o\left(n^{-1}\right)$ then $\sum_{k=1, k\neq j}^{n}\sigma_{n,k}^2 = o(1)$ so $\sum_{k=1, k\neq j}^{n}\left[\frac{\widetilde{\mathbf{B}_n(z)}_{jk}}{\det(\mathbf{B}_n(z))}\right]_{z=1}^{+\circ 2}\sigma_{n,k}^2 = O\left(\sum_{k=1, k\neq j}^{n}\sigma_{n,k}^2\right) = o(1)$. Finally P($iii$) holds by construction and Assumption P(iv) also holds since $\mathbf{A}_n = \mathbf{T}_n^*$.

5 Proof of Expression (5)

Since $\mathbf{T}_n^* = \mathrm{Re}\left(\mathbf{T}_n \right)$, we start expressing the coefficients of the latter matrix.
Since $g(\delta, x) = 1_{\{0 \le x < \pi\delta\}} + 1_{\left\{\pi\left(\frac{3}{2}-\delta\right) < x \le \frac{3\pi}{2}\right\}}$ for $\delta \in (0,1)$ and $x \in [0, 2\pi]$, the coefficients of $\mathbf{T}_n$ satisfy:

$$\begin{aligned} t_{-k}^{(n)} &= \frac{1}{n}\sum_{\ell=0}^{n-1} 1_{\left\{\ell < \frac{n\delta_n}{2}\right\}} e^{2i\pi k\ell/n} + \frac{1}{n}\sum_{\ell=0}^{n-1} 1_{\left\{\left(\frac{3n}{4}-\frac{n\delta_n}{2}\right) < \ell \le \frac{3n}{4}\right\}} e^{2i\pi k\ell/n} \\ &= \frac{1}{n}\sum_{\ell=0}^{\lceil n\delta_n/2\rceil - 1} e^{2i\pi k\ell/n} + \frac{1}{n}\sum_{\ell=\left\lfloor \frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor+1}^{\left\lfloor\frac{3n}{4}\right\rfloor} e^{2i\pi k\ell/n}. \end{aligned}$$

Hence, for $k=0$, $t_0^{(n)} = t_0^{*(n)}$, i.e., $t_0^{(n)} = \frac{\lceil n\delta_n/2\rceil + \left\lfloor\frac{3n}{4}\right\rfloor - \left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor}{n}$ so

$$\begin{aligned} t_0^{(n)} &= \frac{1}{n}\left(\lceil n\delta_n/2\rceil - n\delta_n/2 + \left\lfloor\frac{3n}{4}\right\rfloor - \frac{n\delta_n}{2} - \left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor\right) \\ &= \frac{1}{n}\left(n\delta_n + \left\lceil n/4 + o\left(n^{-1}\right)\right\rceil - n/4 + \left\lfloor\frac{3n}{4}\right\rfloor - \frac{n}{4} - \left\lfloor\frac{3n}{4} - \frac{n}{4} + o\left(n^{-1}\right)\right\rfloor + o\left(n^{-1}\right)\right) \\ &= \delta + (\delta_n - \delta) + \frac{1}{4}\left(\frac{\lceil n/4\rceil - n/4}{n/4} + o\left(n^{-1}\right)\right) - \frac{1}{2}\left(\frac{\left\lfloor\frac{3n}{4}-\frac{n}{4}\right\rfloor - \left\lfloor\frac{3n}{4}\right\rfloor - \frac{n}{4}}{\left\lfloor\frac{3n}{4}\right\rfloor - \frac{n}{4}} + o\left(n^{-1}\right)\right) \end{aligned}$$

yielding $t_0^{(n)} = \delta + (\delta_n - \delta) + O\left(n^{-1}\right)$, and therefore when $n^2(\delta - \delta_n) \to 0$, $t_0^{*(n)} = \delta + O\left(n^{-1}\right)$.

Now, when $k \neq 0$,

$$\begin{aligned} t_{-k}^{(n)} &= \frac{1}{n}\frac{1 - e^{2i\pi k\lceil n\delta_n/2\rceil/n} + e^{2i\pi k\left(\left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor+1\right)/n} - e^{2i\pi k\left(\left\lfloor\frac{3n}{4}\right\rfloor+1\right)/n}}{1 - e^{2i\pi k/n}} \\ &= \frac{1}{n}\frac{e^{\frac{i\pi k\left(\frac{2\lceil n\delta_n/2\rceil}{n}\right)}{2}}\left(e^{-\frac{i\pi k\left(\frac{2\lceil n\delta_n/2\rceil}{n}\right)}{2}} - e^{\frac{i\pi k\left(\frac{2\lceil n\delta_n/2\rceil}{n}\right)}{2}}\right)}{e^{\frac{i\pi k}{n}}\left(e^{-\frac{i\pi k}{n}} - e^{\frac{i\pi k}{n}}\right)} \\ &\quad - \frac{1}{n}\frac{e^{2i\pi k\left(\left\lfloor\frac{3n}{4}\right\rfloor+1\right)/n}\left[1 - e^{-2i\pi k\left(\frac{n\delta_n}{2} + \left(\left\lfloor\frac{3n}{4}\right\rfloor - \frac{n\delta_n}{2} - \left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor\right)\right)/n}\right]}{e^{\frac{i\pi k}{n}}\left(e^{-\frac{i\pi k}{n}} - e^{\frac{i\pi k}{n}}\right)} \end{aligned}$$

so $t_{-k}^{(n)} = \frac{1}{n}\frac{1 - e^{2i\pi k\lceil n\delta_n/2\rceil/n} + e^{2i\pi k\left(\left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor+1\right)/n} - e^{2i\pi k\left(\left\lfloor\frac{3n}{4}\right\rfloor+1\right)/n}}{1 - e^{2i\pi k/n}}$, which yields

$$\begin{aligned} t_{-k}^{(n)} &= \frac{1}{n}\frac{e^{\frac{i\pi k\left(\frac{2\lceil n\delta_n/2\rceil}{n}\right)}{2}}\left(e^{-\frac{i\pi k\left(\frac{2\lceil n\delta_n/2\rceil}{n}\right)}{2}} - e^{\frac{i\pi k\left(\frac{2\lceil n\delta_n/2\rceil}{n}\right)}{2}}\right)}{e^{\frac{i\pi k}{n}}\left(e^{-\frac{i\pi k}{n}} - e^{\frac{i\pi k}{n}}\right)} \\ &\quad - \frac{1}{n}\frac{e^{2i\pi k\left(\left\lfloor\frac{3n}{4}\right\rfloor+1\right)/n}\left[1 - e^{-2i\pi k\left(\frac{n\delta_n}{2} + \left(\left\lfloor\frac{3n}{4}\right\rfloor - \frac{n\delta_n}{2} - \left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor\right)\right)/n}\right]}{e^{\frac{i\pi k}{n}}\left(e^{-\frac{i\pi k}{n}} - e^{\frac{i\pi k}{n}}\right)} \end{aligned}$$

simplifying further,

$$t_{-k}^{(n)} = \frac{1}{n}\frac{e^{\frac{i\pi k\delta_n}{2}\left(\frac{\lceil n\delta_n/2\rceil}{\delta_n n/2}\right)}\sin\left\{\frac{\pi k\delta_n}{2}\frac{\lceil n\delta_n/2\rceil}{\delta_n n/2}\right\}}{e^{\frac{i\pi k}{n}}\sin\frac{\pi k}{n}}$$

$$+\frac{1}{n}\frac{e^{\frac{3i\pi k}{2}\left(1+\frac{\lfloor 3n/4\rfloor-3n/4+1}{3n/4}\right)}e^{-\frac{i\pi k\delta_n}{2}\left(1+\frac{\left\lfloor\frac{3n}{4}\right\rfloor-\frac{n\delta_n}{2}-\left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor}{n\delta_n/2}\right)}\sin\left\{\frac{\pi k\delta_n}{2}\left(1+\frac{\left\lfloor\frac{3n}{4}\right\rfloor-\frac{n\delta_n}{2}-\left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor}{n\delta_n/2}\right)\right\}}{e^{\frac{i\pi k}{n}}\sin\frac{\pi k}{n}}$$

$$=\frac{e^{\frac{i\pi k\delta_n}{2}\left(\frac{\lceil n\delta_n/2\rceil}{\delta_n n/2}\right)}\left(\sin\left\{\frac{\pi k\delta_n}{2}\frac{\lceil n\delta_n/2\rceil}{\delta_n n/2}\right\}\right)}{ne^{\frac{i\pi k}{n}}\sin\frac{\pi k}{n}}$$

$$+\frac{e^{\frac{i\pi k(3-\delta_n)}{2}}e^{\frac{3i\pi k}{2}\frac{\lfloor 3n/4\rfloor-3n/4+1}{3n/4}-\frac{i\pi k\delta_n}{2}\frac{\left\lfloor\frac{3n}{4}\right\rfloor-\frac{n\delta_n}{2}-\left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor}{n\delta_n/2}}\sin\left\{\frac{\pi k\delta_n}{2}\left(1+\frac{\left\lfloor\frac{3n}{4}\right\rfloor-\frac{n\delta_n}{2}-\left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor}{n\delta_n/2}\right)\right\}}{ne^{\frac{i\pi k}{n}}\sin\frac{\pi k}{n}}$$

so, finally,

$$t_{-k}^{(n)} = \frac{\sin\frac{\pi k\delta_n}{2}}{ne^{\frac{i\pi k}{n}}\sin\frac{\pi k}{n}}\left[e^{\frac{i\pi k\delta_n}{2}\left(\frac{\lceil n\delta_n/2\rceil}{\delta_n n/2}\right)}\frac{\sin\left\{\frac{\pi k\delta_n}{2}\frac{\lceil n\delta_n/2\rceil}{\delta_n n/2}\right\}}{\sin\frac{\pi k\delta_n}{2}}+e^{\frac{i\pi k\delta_n}{2}\left(\frac{\lceil n\delta_n/2\rceil}{\delta_n n/2}\right)}\frac{\sin\left\{\frac{\pi k\delta_n}{2}\frac{\lceil n\delta_n/2\rceil}{\delta_n n/2}\right\}}{\sin\frac{\pi k\delta_n}{2}}\right.$$

$$\left.+e^{\frac{i\pi k(3-\delta_n)}{2}}e^{\frac{3i\pi k}{2}\frac{\lfloor 3n/4\rfloor-3n/4+1}{3n/4}-\frac{i\pi k\delta_n}{2}\frac{\left\lfloor\frac{3n}{4}\right\rfloor-\frac{n\delta_n}{2}-\left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor}{n\delta_n/2}}\frac{\sin\left\{\frac{\pi k\delta_n}{2}\left(1+\frac{\left\lfloor\frac{3n}{4}\right\rfloor-\frac{n\delta_n}{2}-\left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor}{n\delta_n/2}\right)\right\}}{\sin\frac{\pi k\delta_n}{2}}\right].$$

Using $e^{i\left(\frac{3\pi k}{2}-x\right)}=(-1)^k\,e^{i\left(\frac{\pi k}{2}-x\right)}$, the previous expression can be rewritten as follows:

$$\begin{aligned}t_{-k}^{(n)} &= \frac{\sin\frac{\pi k\delta_n}{2}}{ne^{\frac{i\pi k}{n}}\sin\frac{\pi k}{n}}\left[e^{\frac{i\pi k\delta_n}{2}}+(-1)^k\,e^{\frac{i\pi k(1-\delta_n)}{2}}\right]+\zeta_k\left(\delta_n,n\right)\\ &=\frac{\delta_n}{2}\frac{\sin_c\frac{\pi k\delta_n}{2}}{e^{\frac{i\pi k}{n}}\sin_c\frac{\pi k}{n}}\left[e^{\frac{i\pi k\delta_n}{2}}+(-1)^k\,e^{\frac{i\pi k(1-\delta_n)}{2}}\right]+\zeta_k\left(\delta_n,n\right),\end{aligned}$$

where

$$\begin{aligned}\zeta_k\left(\delta_n,n\right)=&\left(e^{\frac{i\pi k\delta_n}{2}\left(\frac{\lceil n\delta_n/2\rceil-n\delta_n/2}{n\delta_n/2}\right)}\right)\left(\frac{\sin\left\{\frac{\pi k\delta_n}{2}\frac{\lceil n\delta_n/2\rceil}{n\delta_n/2}\right\}}{\sin\frac{\pi k\delta_n}{2}}-1\right)\\ &-e^{\frac{i\pi k(1-\delta_n)}{2}}\left(e^{\frac{3i\pi k}{2}\frac{\lfloor 3n/4\rfloor-3n/4+1}{3n/4}-\frac{i\pi k\delta_n}{2}\frac{\left\lfloor\frac{3n}{4}\right\rfloor-\frac{n\delta_n}{2}-\left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor}{n\delta_n/2}}\right.\\ &\left.\times\frac{\sin\left\{\frac{\pi k\delta_n}{2}\left(1+\frac{\left\lfloor\frac{3n}{4}\right\rfloor-\frac{n\delta_n}{2}-\left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor}{n\delta_n/2}\right)\right\}}{\sin\frac{\pi k\delta_n}{2}}-1\right).\end{aligned}$$

Therefore

$$
\begin{aligned}
t_{-k}^{(n)} &= \frac{\delta_n}{2}\frac{\sin_c \frac{\pi k \delta_n}{2}}{e^{\frac{i\pi k}{n}}\sin_c \frac{\pi k}{n}} e^{\frac{i\pi k}{4}}\left[e^{-\frac{i\pi k(1/2-\delta_n)}{2}} + (-1)^k e^{\frac{i\pi k(1/2-\delta_n)}{2}}\right] + \zeta_k\left(\delta_n, n\right) \\
&= \delta_n \frac{\sin_c \frac{\pi k \delta_n}{2}}{\sin_c \frac{\pi k}{n}} e^{\frac{i\pi}{2}\left(\left(\frac{1}{2}-\frac{2}{n}\right)k\right)}\left[1_{\{k \text{ odd}\}} e^{-i\frac{\pi}{2}}\sin\frac{\pi k\left(\frac{1}{2}-\delta_n\right)}{2} + 1_{\{k \text{ even}\}}\cos\frac{\pi k\left(\frac{1}{2}-\delta_n\right)}{2}\right] + \zeta_k\left(\delta_n, n\right) \\
&= \delta_n e^{i\frac{\pi k}{4}}\sin_c\frac{\pi k\delta_n}{2}\left[1_{\{k \text{ odd}\}} e^{-i\frac{\pi}{2}}\sin\frac{\pi k\left(\frac{1}{2}-\delta_n\right)}{2} + 1_{\{k \text{ even}\}}\cos\frac{\pi k\left(\frac{1}{2}-\delta_n\right)}{2}\right]\left(\frac{e^{-i\pi\frac{k}{n}}}{\sin_c\frac{\pi k}{n}}\right) + \zeta_k\left(\delta_n, n\right) \\
&= \delta_n e^{i\frac{\pi k}{4}}\sin_c\frac{\pi k\delta_n}{2}\left[1_{\{k \text{ odd}\}} e^{-i\frac{\pi}{2}}\sin\frac{\pi k\left(\frac{1}{2}-\delta_n\right)}{2} + 1_{\{k \text{ even}\}}\cos\frac{\pi k\left(\frac{1}{2}-\delta_n\right)}{2}\right] + \xi_k\left(\delta_n, n\right) + \zeta_k\left(\delta_n, n\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_k\left(\delta_n, n\right) = {} & \delta_n e^{i\frac{\pi k}{4}}\sin_c\frac{\pi k\delta_n}{2}\left[1_{\{k \text{ odd}\}} e^{-i\frac{\pi}{2}}\sin\frac{\pi k\left(\frac{1}{2}-\delta_n\right)}{2} + 1_{\{k \text{ even}\}}\cos\frac{\pi k\left(\frac{1}{2}-\delta_n\right)}{2}\right] \\
& \times\left(\frac{e^{-i\pi\frac{k}{n}}}{\sin_c\frac{\pi k}{n}} - 1\right).
\end{aligned}
$$

It remains to be shown that both $\xi_k\left(\delta_n, n\right)$ and $\zeta_k\left(\delta_n, n\right)$ are $O\left(n^{-1}\right)$. We use the fact that, as $x \to 0$, $\sin x = x + O\left(x^3\right)$, $\sin_c x = 1 + O\left(x^2\right)$ and, when $\sin a \neq 0$, $\sin(a+x) = \sin a + x\cos a + O\left(x^2\right)$ and $\sin_c(a+x) = \sin_c a + O(x)$. Hence

$$
\begin{aligned}
\xi_k\left(\delta_n, n\right) &= \left(\frac{1}{2} + O\left(\frac{1}{2}-\delta_n\right)\right) e^{i\frac{\pi(k-2)}{4}}\left(\sin_c\left(\frac{\pi k}{4}\right) + O\left(k\left(\frac{1}{2}-\delta_n\right)\right)\right) \\
&\quad \times\left[\frac{\pi k\left(\frac{1}{2}-\delta_n\right)}{2} + O\left(k\left(\frac{1}{2}-\delta_n\right)^3\right)\right]\left(\frac{1+O\left(\frac{k}{n}\right)}{1+O\left(\frac{k}{n}\right)} - 1\right) \\
&= \frac{1}{2}e^{i\frac{\pi(k-2)}{4}}\sin_c\left(\frac{\pi k}{4}\right)\left(1 + O\left(\frac{1}{2}-\delta_n\right)\right)\times\left[\frac{\pi k\left(\frac{1}{2}-\delta_n\right)}{2} + O\left(k\left(\frac{1}{2}-\delta_n\right)^3\right)\right]\left(O\left(\frac{k}{n}\right)\right) \\
&= O\left(\frac{k^2}{n}\left(\frac{1}{2}-\delta_n\right)\right) = O\left(n\left(\frac{1}{2}-\delta_n\right)\right),
\end{aligned}
$$

and therefore when $n^2\left(1/2-\delta_n\right) \to 0$, $\xi_k\left(\delta_n, n\right) = o\left(n^{-1}\right)$ while

$$
\begin{aligned}
\zeta_k\left(\delta_n, n\right) = {} & \left(e^{\frac{i\pi k\delta_n}{2}\left(\frac{\lceil n\delta_n/2\rceil - n\delta_n/2}{n\delta_n/2}\right)}\right)\left(\frac{\sin\left\{\frac{\pi k\delta_n}{2}\frac{\lceil n\delta_n/2\rceil}{n\delta_n/2}\right\}}{\sin\frac{\pi k\delta_n}{2}} - 1\right) \\
& - e^{\frac{i\pi k(1-\delta_n)}{2}}\left(e^{\frac{3i\pi k}{2}\frac{\lfloor 3n/4\rfloor - 3n/4+1}{3n/4} - \frac{i\pi k\delta_n}{2}\frac{\left\lfloor\frac{3n}{4}\right\rfloor - \frac{n\delta_n}{2} - \left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor}{n\delta_n/2}}\right. \\
& \left.\times\frac{\sin\left\{\frac{\pi k\delta_n}{2}\left(1 + \frac{\left\lfloor\frac{3n}{4}\right\rfloor - \frac{n\delta_n}{2} - \left\lfloor\frac{3n}{4}-\frac{n\delta_n}{2}\right\rfloor}{n\delta_n/2}\right)\right\}}{\sin\frac{\pi k\delta_n}{2}} - 1\right).
\end{aligned}
$$

We note the following expressions

$$e^{\frac{i\pi k\delta_n}{2}\left(\frac{\lceil n\delta_n/2\rceil - n\delta_n/2}{n\delta_n/2}\right)} = 1 + O\left(n^{-1}\right)$$
$$\sin\left\{\frac{\pi k\delta_n}{2}\frac{\lceil n\delta_n/2\rceil}{n\delta_n/2}\right\} = \sin\frac{\pi k\delta_n}{2} + O\left(n^{-1}\right)$$
$$e^{\frac{3i\pi k}{2}\frac{\lfloor 3n/4\rfloor - 3n/4+1}{3n/4} - \frac{i\pi k\delta_n}{2}\frac{\left\lfloor\frac{3n}{4}\right\rfloor - \frac{n\delta_n}{2} - \left\lfloor\frac{3n}{4} - \frac{n\delta_n}{2}\right\rfloor}{n\delta_n/2}} = 1 + O\left(n^{-1}\right)$$
$$\sin\left\{\frac{\pi k\delta_n}{2}\left(1 + \frac{\left\lfloor\frac{3n}{4}\right\rfloor - \frac{n\delta_n}{2} - \left\lfloor\frac{3n}{4} - \frac{n\delta_n}{2}\right\rfloor}{n\delta_n/2}\right)\right\} = \sin\frac{\pi k\delta_n}{2} + O\left(n^{-1}\right).$$

Hence, $\zeta_k\left(\delta_n, n\right) = \left(1 + O\left(n^{-1}\right)\right)\left(\frac{\sin\frac{\pi k\delta_n}{2} + O\left(n^{-1}\right)}{\sin\frac{\pi k\delta_n}{2}} - 1\right) - e^{\frac{i\pi k(1-\delta_n)}{2}}\left(\left[1 + O\left(n^{-1}\right)\right]\frac{\sin\frac{\pi k\delta_n}{2} + O\left(n^{-1}\right)}{\sin\frac{\pi k\delta_n}{2}} - 1\right)$, i.e., $\zeta_k\left(\delta_n, n\right) = O\left(n^{-1}\right)$. Now, $t_{-k}^{*(n)} = \text{Re}\left(t_{-k}^{(n)}\right)$ so $t_{-k}^{*(n)} = \delta_n \sin_c\frac{\pi k\delta_n}{2}\left[1_{\{k \text{ odd}\}}\sin\frac{\pi k}{4}\sin\frac{\pi k\left(\frac{1}{2}-\delta_n\right)}{2} + 1_{\{k \text{ even}\}}\cos\frac{\pi k}{4}\cos\frac{\pi k\left(\frac{1}{2}-\delta_n\right)}{2}\right] + O\left(n^{-1}\right)$. Notice that $k\left(\frac{1}{2} - \delta_n\right) = o\left(n^{-1}\right)\ \forall k < n$, hence for k odd, $t_{-k}^{*(n)} = O\left(n^{-1}\right)$. When k is even, we need to consider the cases where there exists an odd integer m such that $k = 4m$ or $k = 4m+2$. First if $k = 4m$ then $\sin\frac{\pi k\delta_n}{2} = \sin 2\pi m\delta_n = O\left(m\left(\frac{1}{2} - \delta_n\right)\right) = o\left(n^{-1}\right)$ and if $k = 4m+2$, then $\cos\frac{\pi k}{4} = \cos\left(m\pi + \frac{\pi}{2}\right) = 0$. Hence for all k such that $0 < |k| < n$,

$$t_{-k}^{*(n)} = O\left(n^{-1}\right),$$

which concludes the proof of Expression (5).

References

Whittaker, J. M. (1930-31). The absolute summability of Fourier series. *Proceedings of the Edinburgh Mathematical Society 2*(2), 1–5.